

An Algebra Containing the Two-Sided Convolution Operators

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Abstract

We present an intrinsically defined algebra of operators containing the right and left invariant Calderón-Zygmund operators on a stratified group. The operators in our algebra are pseudolocal and bounded on L^p ($1 < p < \infty$). This algebra provides an example of an algebra of singular integrals that falls outside of the classical Calderón-Zygmund theory.

1 Introduction

Let G be a stratified Lie group. That is, G is connected, simply connected, and its Lie Algebra \mathfrak{g} may be decomposed $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, where $[V_1, V_k] = V_{k+1}$ for $1 \leq k < m$ and $[V_1, V_m] = 0$. The Calderón-Zygmund theory for left (or right) invariant convolution operators on G is well-known (see [Ste93], and Section 3 for a review). Given a distribution kernel K as in Definition 3.2 one obtains two “Calderón-Zygmund singular integral operators”:

$$\text{Op}_L(K)f := f * K$$

$$\text{Op}_R(K)f := K * f$$

The operators of the form $\text{Op}_L(K)$ form an algebra ($\text{Op}_L(K_1)\text{Op}_L(K_2) = \text{Op}_L(K_1 * K_2)$), are bounded on L^p ($1 < p < \infty$), and are pseudolocal. The same is true for operators of the form $\text{Op}_R(K)$. Also, if we consider:

$$\text{Op}_L(K_1)\text{Op}_R(K_2)f = (K_2 * f) * K_1 = K_2 * (f * K_1) = \text{Op}_R(K_2)\text{Op}_L(K_1)f$$

we see that $\text{Op}_L(K_1)$ and $\text{Op}_R(K_2)$ commute.

Hence, it follows that:

$$\text{Op}_L(K_1)\text{Op}_R(K_2)\text{Op}_L(K_3)\text{Op}_R(K_4) = \text{Op}_L(K_1 * K_3)\text{Op}_R(K_4 * K_2)$$

and so operators of the form $\text{Op}_L(K_1)\text{Op}_R(K_2)$ are closed under composition. It is also evident that they are bounded on L^p ($1 < p < \infty$) and are pseudolocal.

The main goal of this paper is to present an algebra of operators, which contains operators of the form $\text{Op}_L(K_1)\text{Op}_R(K_2)$, and such that the operators in this algebra are bounded on L^p ($1 < p < \infty$), and are pseudolocal. Moreover,

the algebra will contain the so-called two-sided convolution operators, of the form:

$$Tf(x) = \int K(y, z) f(z^{-1}xy^{-1}) dydz \quad (1)$$

where K is a product kernel (see Section 3.2). This algebra provides a naturally occurring example that falls outside of the classical Calderón-Zygmund paradigm.

Operators that fall outside of the classical Calderón-Zygmund paradigm often arise in the construction of parametrices of hypoelliptic operators which are not maximally subelliptic. In fact, one of the original motivations for the present paper was the form of the parametrix constructed in [Str07] for Kohn's example of a sum of squares of complex vector fields, whose commutators span the tangent space at each point, and such that the sum of squares is hypoelliptic but not subelliptic ([Koh05]). The parametrix is constructed from compositions of left and right convolution operators on the three dimensional Heisenberg group, and is therefore closely related to the algebra discussed in this paper. It is our hope that the work in this paper will help to motivate the proper algebras to use in other problems, where the Calderón-Zygmund theory is no longer applicable (for instance, as in [NS06]).

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2 Statement of Results

Recall, G is a stratified group (for a background on such groups see [Fol75]), and as such, the lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, with the V_j satisfying the relations in the introduction.

Fix a basis $X^{(1)}, \dots, X^{(n)}$ for V_1 , thought of as elements of the tangent space to the identity in G . Then we can think of each $X^{(j)}$ as either a right invariant or a left invariant vector field, call them $X_R^{(j)}$ and $X_L^{(j)}$ respectively. From here, we get the left and right gradients:

$$\nabla_L = (X_L^{(1)}, \dots, X_L^{(n)})$$

$$\nabla_R = (X_R^{(1)}, \dots, X_R^{(n)})$$

Our definitions will be in terms of ∇_L and ∇_R , but will not depend in any essential way on the specific choice of basis of V_1 . Throughout this paper, we will use ordered multi-index notation. Thus, for a finite sequence s of numbers

$1, \dots, n$, we define $|s|$ to be the length of the sequence, and ∇_L^s, ∇_R^s in the usual way. So that, for instance:

$$\nabla_L^{(i,j,k)} = X_L^{(i)} X_L^{(j)} X_L^{(k)}$$

and $|(i, j, k)| = 3$.

For $\epsilon \geq 0$, let ρ_ϵ^L denote the Carnot-Carathéodory distance on G associated to the vector fields $\{\nabla_L, \epsilon \nabla_R\}$ and ρ_ϵ^R the one associated to the vector fields $\{\nabla_R, \epsilon \nabla_L\}$ (see Section 4 and references therein for background on such metrics). Let $B_\epsilon^L(x, \delta)$ denote the ball centered at x of radius δ in the ρ_ϵ^L metric, and $V_\epsilon^L(x, \delta)$ its volume. Similarly, we define B_ϵ^R and V_ϵ^R in terms of ρ_ϵ^R .

Definition 2.1. For $r_R \geq r_L > 0$, we say $\phi \in C^\infty(G)$ is a normalized r_L, r_R bump function of order M centered at $x \in G$ if ϕ is supported in $B_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)$, and $\forall |\alpha| + |\beta| \leq M$,

$$\left| \nabla_L^\alpha \nabla_R^\beta \phi \right| \leq \frac{r_L^{|\alpha|} r_R^{|\beta|}}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$$

When $r_L \geq r_R$, we replace $V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)$ with $V_{\frac{r_R}{r_L}}^R\left(x, \frac{1}{r_R}\right)$.

We define (for $0 < r_L \leq r_R$)

$$\begin{aligned} B(r_L, r_R, N_L, N_R, m, x, y) \\ = r_L^{N_L} r_R^{N_R} \left(1 + r_L \rho_{\frac{r_L}{r_R}}^L(x, y) \right)^{-m} \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L} + \rho_{\frac{r_L}{r_R}}^L(x, y)\right)} \end{aligned}$$

and when $r_R \geq r_L$, we reverse the roles of r_L and r_R and of the left and right vector fields. Before we define our algebra rigorously, let us write the definition while being a little loose with quantifiers. We say that $T \in \mathcal{A}$ if for every $m \geq 0$, and for every $\phi_{r_L^{(1)}, r_R^{(1)}}^x$ normalized $r_L^{(1)}, r_R^{(1)}$ bump functions centered at x and every $\phi_{r_L^{(2)}, r_R^{(2)}}^y$ normalized $r_L^{(2)}, r_R^{(2)}$ bump functions centered at y (we suppress the order for the moment), we have:

$$\begin{aligned} \left| \left\langle \phi_{r_L^{(1)}, r_R^{(1)}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{r_L^{(2)}, r_R^{(2)}}^y \right\rangle_{L^2} \right| \\ \leq CB \left(r_L^{(1)} \wedge r_L^{(2)}, r_R^{(1)} \wedge r_R^{(2)}, |\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2|, m, x, y \right) \end{aligned}$$

where $|\alpha_1| + |\alpha_2|$ and $|\beta_1| + |\beta_2|$ must be sufficiently large depending on m , and C is uniform in the choice of normalized bump function, $r_L^{(1)}, r_R^{(1)}, r_L^{(2)}, r_R^{(2)}$, and in x, y . Here, and in the rest of the paper, $a \wedge b$ denotes the minimum of a and b , while $a \vee b$ denotes the maximum. Rigorously, our definition is:

Definition 2.2. We define \mathcal{A} to be the set of those operators $T : \mathcal{S}(G) \rightarrow \mathcal{S}(G)'$, such that for all $m \geq 0$, there exists N_0 , such that for all $N_L, N_R \geq N_0$ there exists $C > 0$, $M \in \mathbb{N}$ such that for all $x, y \in G$, all $r_L^{(1)}, r_L^{(2)}, r_R^{(1)}, r_R^{(2)} > 0$, all $\phi_{r_L^{(1)}, r_R^{(1)}}^x$ normalized $r_L^{(1)}, r_R^{(1)}$ bump functions centered at x of order M , all $\phi_{r_L^{(2)}, r_R^{(2)}}^y$ normalized $r_L^{(2)}, r_R^{(2)}$ bump functions centered at y of order M , and all $|\alpha_1| + |\alpha_2| = N_L$, $|\beta_1| + |\beta_2| = N_R$, we have:

$$\begin{aligned} & \left| \left\langle \phi_{r_L^{(1)}, r_R^{(1)}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{r_L^{(2)}, r_R^{(2)}}^y \right\rangle_{L^2} \right| \\ & \leq CB \left(r_L^{(1)} \wedge r_L^{(2)}, r_R^{(1)} \wedge r_R^{(2)}, N_L, N_R, m, x, y \right) \end{aligned} \quad (2)$$

Remark 2.3. We will see *a posteriori* that $N_0 = Q + m + 1$ will work. See Remark 7.2.

We will show:

1. The operators in \mathcal{A} are the same as those in \mathcal{A}' (defined below; see Section 7).
2. The operators in \mathcal{A}' extend uniquely to bounded operators on L^p , $1 < p < \infty$ (Section 8).
3. If $T \in \mathcal{A}$, then $T^* \in \mathcal{A}$, where T^* denotes the L^2 adjoint of T (Remark 2.8).
4. The operators in \mathcal{A}' form an algebra (Remark 2.7).
5. The operators in \mathcal{A}' are pseudolocal (Section 9).
6. Two-sided convolution operators (and therefore the right and left Calderón-Zygmund operators) are in \mathcal{A}' (Corollary 6.7).

Our main technical result is that the operators in \mathcal{A} are the same as those in \mathcal{A}' . To define \mathcal{A}' , we need a preliminary definition.

Definition 2.4. We say that $\phi(x, z) \in C^\infty(G \times G)$ is an r_L, r_R elementary kernel if, for every m and every $\alpha_1, \beta_1, \alpha_2, \beta_2$, there exists a $C = C(m, \alpha_1, \alpha_2, \beta_1, \beta_2)$ such that

$$\begin{aligned} & \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} \phi(x, z) \right| \\ & \leq CB(r_L, r_R, |\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2|, m, x, z) \end{aligned} \quad (3)$$

and, for every $N_1, N_2, N_3, N_4 \in \mathbb{N}$, and every $|\alpha_1| = N_1$, $|\alpha_2| = N_2$, $|\beta_1| = N_3$, $|\beta_2| = N_4$, there exist functions $\psi_{\alpha_1, \alpha_2, \beta_1, \beta_2} \in C^\infty(G \times G)$ such that

$$\phi = r_L^{-N_1 - N_2} r_R^{-N_3 - N_4} \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} \psi_{\alpha_1, \alpha_2, \beta_1, \beta_2}$$

and the ψ satisfy (3) with different constants. Finally, we say E is an r_L, r_R elementary operator if the Schwartz kernel of E is an elementary kernel.

For each r_L, r_R , Definition 2.4 implicitly defines a family of seminorms of the elementary kernels (ie, the least possible C in (3), and the least possible C obtained from all choices of ψ , etc.). If $L : C^\infty(G \times G) \rightarrow C^\infty(G \times G)$ is a linear map that takes r_L, r_R elementary kernels to r_L, r_R elementary kernels, continuously, it makes sense to ask if it does so *uniformly* in r_L, r_R , since we may order the semi-norms consistently as r_L and r_R vary.

Definition 2.5. We define \mathcal{A}' to be those operators $T : \mathcal{S}_0(G) \rightarrow \mathcal{S}_0(G)$ such that for each r_L, r_R , and every E an r_L, r_R elementary operator, TE is an r_L, r_R elementary operator, and this map is uniformly continuous in r_L, r_R . Here \mathcal{S}_0 is the set of Schwartz functions, all of whose moments vanish.

Remark 2.6. The operators in \mathcal{A}' are *a priori* defined only on \mathcal{S}_0 . To see that they are the same as those in \mathcal{A} , we first extend them as bounded operators on L^2 , and then prove that the extended operator is in \mathcal{A} .

Remark 2.7. It is evident that if $T_1, T_2 \in \mathcal{A}'$ then $T_1 T_2 \in \mathcal{A}'$. We will show that the operators in \mathcal{A}' are the same as those in \mathcal{A} , and therefore \mathcal{A} forms an algebra.

The operators in \mathcal{A} may be thought of as “smoothing of order 0.” In Section 10 we define the analogous concept of operators which are smoothing of other orders. In Section 10 we also discuss an alternative to Definition 2.2, and why a definition like Definition 2.2 seems to be necessary.

Remark 2.8. Definitions 2.2 and 2.4 may not seem to be symmetric (eg, if $T \in \mathcal{A}$ is $T^* \in \mathcal{A}$? and if E is an r_L, r_R elementary kernel, is E^* ?), however they are. Indeed, despite the fact that $B(\cdot, \cdot, \cdot, \cdot, \cdot, x, y)$ is not symmetric in x and y , it follows from the results in Section 4 that there exists a $C > 0$ such that:

$$\frac{1}{C} B(\cdot, \cdot, \cdot, \cdot, \cdot, y, x) \leq B(\cdot, \cdot, \cdot, \cdot, \cdot, x, y) \leq C B(\cdot, \cdot, \cdot, \cdot, \cdot, y, x)$$

Some words on notation. When we refer to the “unit ball”, we are always referring to the set $\{x : \|x\| < 1\}$, where $\|\cdot\|$ is defined in Section 3. $A \lesssim B$ will always mean $A \leq CB$, where C is some constant, independent of any relevant parameters, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. Sometimes we will have a sum of positive numbers of the form

$$\sum_{n \geq 0} a_n$$

and we will have

$$\sum_{n \geq 0} a_n \lesssim \sum_{n \geq 0} r^n a_0$$

for some r , $0 < r < 1$. In this case we will say the series $\sum_{n \geq 0} a_n$ falls off geometrically or even “is geometric,” and we will use the fact that in this case $\sum_{n \geq 0} a_n \approx a_0$.

3 Calderón-Zygmund Operators

In this section, we will remind the reader of the standard theory of Calderón-Zygmund convolution operators on G . Our goal is three-fold: first to fix notation, second to present these concepts in a few different ways, each of which will be useful in understanding our more complicated algebra, and finally we will need these characterizations to show that these Calderón-Zygmund operators are in our algebra.

Recall, G is a stratified group, and so, as in the introduction, the Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_n$, where the V_j satisfy the relations in the introduction. The exponential map $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism. We define dilations of \mathfrak{g} , which for $r > 0$ are given by $r \cdot X = r^j X$ for $X \in V_j$. These dilations induce automorphisms of G by $re^X = e^{r \cdot X}$. If we identify G with \mathfrak{g} via the exponential map, Lebesgue measure becomes Haar measure for G , and $d(rx) = r^Q dx$ for some $Q \in \mathbb{N}$. We call Q the “homogeneous dimension” of G . For a function $\phi : G \rightarrow \mathbb{C}$ and $r > 0$, we define $\phi^{(r)}(x) = r^Q \phi(rx)$. Let $\|\cdot\| : G \rightarrow \mathbb{R}^+$ be a smooth homogeneous norm. See [Fol75] for a more in depth discussion.

For a background on the material presented here, see [Ste93] and [NRS01]. Indeed, we will be following the presentation of “product kernels” from [NRS01] later in this section.

Definition 3.1. A k -normalized bump function on G is a C^k function supported on the unit ball with C^k norm bounded by 1. The definitions that follow turn out to not depend in any essential way on k , and so we shall speak of *normalized bump functions*, thereby suppressing the dependence on k .

Definition 3.2. A Calderón-Zygmund kernel on G , is a distribution K on G , which coincides with a C^∞ function away from 0, and satisfies:

1. (Differential inequalities) For each ordered multi-index α , there is a constant C_α so that

$$|\nabla_L^\alpha K(x)| \leq C_\alpha \|x\|^{-Q-|\alpha|}$$

one may, equivalently, use ∇_R in place of ∇_L .

2. (Cancellation conditions) Given any normalized bump function ϕ , and any $r > 0$,¹

$$\int K(x) \phi(rx) dx$$

is bounded independent of ϕ and r .

Proposition 3.3. Let K be a distribution on G . Then, K is a Calderón-Zygmund kernel if and only if there exists a sequence $\{\phi_j\}_{j \in \mathbb{Z}} \subset \mathcal{S}_0$, forming a bounded subset of \mathcal{S}_0 , such that

$$K = \sum_{j \in \mathbb{Z}} \phi_j^{(2^j)}$$

¹We will abuse notation and write the pairing between distributions and test functions as an integral.

where this sum is taken in distribution (any such sum converges in distribution). In this case,

$$\text{Op}_L(K) = \sum_{j \in \mathbb{Z}} \text{Op}_L\left(\phi_j^{(2^j)}\right) \quad (4)$$

where this sum is taken in the strong operator topology as bounded operators on L^p ($1 < p < \infty$). In particular, $\text{Op}_L(K)$ is a bounded operator on L^p ($1 < p < \infty$). In addition, (4) converges in the topology of bounded convergence as operators $\mathcal{S}_0 \rightarrow \mathcal{S}_0$. All of the above can be done uniformly over a bounded subset of Calderón-Zygmund kernels. All of the above holds for $\text{Op}_R(K)$ as well.

Proof. This result is essentially contained in the proofs of Theorem 2.2.1, Theorem 2.6.1, and Proposition 2.7.1 of [NRS01]. The only part not appearing in that paper is the convergence in the topology of bounded convergence. This follows in a manner completely analogous to Theorem 6.6. We leave the details to the interested reader. \square

Theorem 3.4. *Let $T : \mathcal{S}_0(G) \rightarrow C_0^\infty(G)'$. Then, $T = \text{Op}_L(K)$ (when restricted to \mathcal{S}_0), where K is a Calderón-Zygmund kernel, if and only if for every $\phi \in \mathcal{S}_0$ and every $r > 0$,*

$$\text{Op}_L(K) \text{Op}_L\left(\phi^{(r)}\right) = \text{Op}_L\left(\psi_r^{(r)}\right)$$

where $\psi_r \in \mathcal{S}_0$, and as ϕ ranges over a bounded set in \mathcal{S}_0 , and r ranges over $r > 0$, we have that ψ_r ranges over a bounded set in \mathcal{S}_0 .

We defer the proof to Section 3.1. Theorem 3.4 should be interpreted in the following way: we think of operators of the form $\text{Op}_L(\phi^{(r)})$, with $\phi \in \mathcal{S}_0$, as our “ r elementary operators” in analogy with Definition 2.4. Theorem 3.4 simply says that T is a Calderón-Zygmund operator if and only if composition with T takes r elementary operators to r elementary operators uniformly, in analogy with Definition 2.5.

We now turn to an equivalent way of considering Calderón-Zygmund operators that is analogous to Definition 2.2. Let K be a Calderón-Zygmund kernel, let $T = \text{Op}_L(K)$, and let ϕ, ψ be normalized bump functions. Define:

$$\phi_r^x(y) = r^Q \phi(r(y^{-1}x))$$

and similarly for ψ_r^x . The cancellation condition of Definition 3.2 shows that $|T\phi_r^{x_0}| \lesssim r^Q$, on $\|x^{-1}x_0\| \leq 2r^{-1}$. Combining this with the growth condition, one sees:

$$|T\phi_r^{x_0}(x)| \lesssim \begin{cases} r^Q & \text{if } \|x^{-1}x_0\| \leq r^{-1}, \\ \|x^{-1}x_0\|^{-Q} & \text{if } \|x^{-1}x_0\| \geq r^{-1}. \end{cases} \quad (5)$$

Conversely, it is clear that Equation (5) implies the cancellation condition of Definition 3.2. To see that it also implies the growth condition (where there are no derivatives involved), merely choose ϕ so that as $r \rightarrow \infty$, $\phi_r^{x_0} \rightarrow \delta_{x_0}$.

Now suppose $s > r$, we see from (5),

$$\begin{aligned}
|\langle \psi_s^{x_1}, T \phi_r^{x_0} \rangle_{L^2}| &\lesssim \begin{cases} r^Q & \text{if } \|x_1^{-1} x_0\| \leq r^{-1}, \\ \|x_1^{-1} x_0\|^{-Q} & \text{if } \|x_1^{-1} x_0\| \geq r^{-1} \end{cases} \\
&\lesssim \frac{1}{\left(\frac{1}{r} + \|x_1^{-1} x_0\|\right)^Q} \\
&\lesssim \frac{1}{V\left(x_0, \frac{1}{r} + \|x_1^{-1} x_0\|\right)}
\end{aligned} \tag{6}$$

where $V(x, \delta)$ denotes the volume of $\{y : \|y^{-1}x\| < \delta\}$. So we see that (6) is implied by (5). The converse is true as well, as can be seen by taking ψ such that $\psi_s^{x_1} \rightarrow \delta_{x_1}$ as $s \rightarrow \infty$. From these considerations, the following theorem follows easily:

Theorem 3.5. *Suppose $T : \mathcal{S} \rightarrow C_0^\infty(G)'$, and is left invariant. Then, as operators on \mathcal{S}_0 , $T = \text{Op}_L(K)$ where K is a Calderón-Zygmund kernel if and only if for all m , and all ϕ and ψ normalized bump functions, and all α, β ordered multi-indices such that $|\alpha| + |\beta|$ is sufficiently large depending on m , we have:*

$$\left| \left\langle \psi_s^{x_1}, \nabla_L^\alpha T \nabla_L^\beta \phi_r^{x_0} \right\rangle_{L^2} \right| \lesssim (1 + t \|x_1^{-1} x_0\|)^{-m} \frac{t^{|\alpha|+|\beta|}}{V\left(x_0, \frac{1}{t} + \|x_1^{-1} x_0\|\right)} \tag{7}$$

where $t = s \wedge r$, and the bound is uniform in s, r, x_0, x_1 , and choices of normalized bump functions. This is analogous to Definition 2.2.

Remark 3.6. We will see later that using the cancellation condition on both sides simultaneously as in (7) seems to be necessary in our situation. See Section 10.

3.1 Proof of Theorem 3.4

Lemma 3.7. *Given $\phi \in \mathcal{S}_0$, there exists $\psi \in \mathcal{S}_0^n$ such that*

$$\phi = \nabla_L \cdot \psi \tag{8}$$

Moreover, for each continuous semi-norm $|\cdot|$ on \mathcal{S}_0 , the infimum over all such ψ of $\sum |\psi_j|$ is a continuous semi-norm on \mathcal{S}_0 .

For $\phi \in \mathcal{S}$, we say $\phi \in \mathcal{S}_{(m)}$ if ϕ can be written as in (8) with $\psi \in \mathcal{S}_{(m-1)}^n$; where $\mathcal{S}_{(0)} = \mathcal{S}$. Then, $\mathcal{S}_0 = \cap_m \mathcal{S}_{(m)}$.

Proof. This lemma is well known. \square

Lemma 3.8. *Suppose $\phi_1, \phi_2 \in \mathcal{S}_0$, then*

$$\phi_1^{(2^j)} * \phi_2^{(2^k)} = 2^{-|j-k|} \psi^{(2^l)}$$

where l can be either $j \wedge k$ or $j \vee k$, $\psi \in \mathcal{S}_0$, and when ϕ_1, ϕ_2 vary over a bounded set of \mathcal{S}_0 , and j, k vary over \mathbb{Z} , ψ varies over a bounded set of \mathcal{S}_0 .

Proof. We prove the result first in the case $l = j \wedge k$ and $j \leq k$, the case $k < j$ is similar. Note that:

$$\phi_1^{(2^j)} * \phi_2^{(2^k)}(x) = 2^{(j+k)Q} \int \phi_1(2^j(xy^{-1})) \phi_2(2^k y) dy$$

and so replacing x with $2^{-j}x$ and doing a change of variables $u = 2^j y^{-1}$, we see that we can just prove the lemma for $j = 0$, $k \geq 0$.

Writing $\phi_2 = \nabla_R \cdot \phi'_2$ as in Lemma 3.7 (using right invariant vector fields, instead of left), we see that

$$\begin{aligned} \phi_1 * \phi_2^{(2^k)} &= 2^{kQ-k} \int \phi_1(xy^{-1}) \nabla_R \cdot \phi'_2(2^k y) dy \\ &= 2^{kQ-k} \int (\nabla_R \phi_1)(xy^{-1}) \cdot \phi'_2(2^k y) dy \end{aligned}$$

Repeating this process Q more times, we see:

$$\phi_1 * \phi_2^{(2^k)} = 2^{-k} \sum_m \int \psi_{1,m}(xy^{-1}) \psi_{2,m}(2^k y) dy \quad (9)$$

where m ranges over a finite set, and the $\psi_{1,m}, \psi_{2,m} \in \mathcal{S}_0$ (and range over a bounded set as ϕ_1, ϕ_2 do).

To see that $\psi_1 * \left(2^{-kQ} \psi_2^{(2^k)}\right)$ is rapidly decreasing (independent of k), we now need to merely apply the fact that if f_1 and f_2 are two bounded rapidly decreasing functions, then $f_1 * f_2$ is rapidly decreasing (note that $\psi_2(2^k y)$ decreases faster than $\psi_2(y)$).

Since, $\nabla_R \left(\phi_1 * \phi_2^{(2^k)}\right) = (\nabla_R \phi_1) * \phi_2$, we see that $\phi_1 * \phi_2^{(2^k)} \in \mathcal{S}$. To see it is really in \mathcal{S}_0 , we use the fact that $\phi_1 = \nabla_R \cdot \phi'_1$, and therefore,

$$\phi_1 * \phi_2^{(2^k)} = (\nabla_R \cdot \phi'_1) * \phi_2^{(2^k)} = \nabla_R \cdot \left(\phi'_1 * \phi_2^{(2^k)}\right)$$

repeating this process and applying Lemma 3.7, completes the proof of the case $l = j \wedge k$.

Turning to the case when $l = j \vee k$, we again assume $j = 0$ and now assume $k > 0$, the other cases being similar. A computation similar to the one leading up to (9) shows that

$$\phi_1 * \phi_2^{(2^k)} = 2^{-Nk} \sum_m \int \psi_{1,m}(xy^{-1}) \psi_{2,m}(2^k y) dy \quad (10)$$

for any fixed N , where the $\psi_{1,m}, \psi_{2,m}$ are as above. Thus, if one wishes to show that

$$\|2^k x\|^N \phi_1 * \phi_2^{(2^k)} \quad (11)$$

is bounded by $C2^{(Q-1)k}$, one merely needs to apply (10) and use the fact that

$$\int \psi_{1,m}(xy^{-1}) \psi_{2,m}(2^k y)$$

is rapidly decreasing, as shown above. In fact, we get the stronger result that (11) is bounded independent of k . Derivatives work as before, yielding the result. \square

Proof of Theorem 3.4. First, suppose that $T = \text{Op}_L(K)$ where K is a Calderón-Zygmund kernel. We prove the result for $r = 2^l$ for some $l \in \mathbb{Z}$. The more general result follows from this by moving r to the closest such 2^l , via replacing ϕ by $\phi(\frac{r}{2^l}x)$.

Applying Proposition 3.3, we may write:

$$\text{Op}_L(K) = \sum_{k \in \mathbb{Z}} \text{Op}_L\left(\psi_k^{(2^k)}\right)$$

with the $\psi_k \in \mathcal{S}_0$ uniformly in k (even as j varies). We now apply Lemma 3.8 to see:

$$\begin{aligned} \text{Op}_L(K) \text{Op}_L\left(\phi^{(2^j)}\right) &= \sum_{k \in \mathbb{Z}} \text{Op}_L\left(\psi_k^{(2^k)}\right) \text{Op}_L\left(\phi^{(2^j)}\right) \\ &= \sum_{k \in \mathbb{Z}} \text{Op}_L\left(\phi^{(2^j)} * \psi_k^{(2^k)}\right) \\ &= \sum_{k \in \mathbb{Z}} 2^{-|k-j|} \text{Op}_L\left(\phi_k^{(2^j)}\right) \\ &=: \text{Op}_L\left(\phi_0^{(2^j)}\right) \end{aligned}$$

where $\phi_0 \in \mathcal{S}_0$ ranges over a bounded set as the relevant variables change.

Conversely, suppose T , satisfies the conditions on the theorem. We wish to show that $T = \text{Op}_L(K)$, where K is a Calderón-Zygmund operator. We know that I (the identity) is a Calderón-Zygmund operator, and therefore, by Proposition 3.3:

$$I = \sum_{k \in \mathbb{Z}} \text{Op}_L\left(\phi_k^{(2^k)}\right)$$

with the convergence in the topology of bounded convergence as operators $\mathcal{S}_0 \rightarrow \mathcal{S}_0$. Hence,

$$\begin{aligned} T &= TI \\ &= \sum_{k \in \mathbb{Z}} T \text{Op}_L\left(\phi_k^{(2^k)}\right) \\ &= \sum_{k \in \mathbb{Z}} \text{Op}_L\left(\psi_k^{(2^k)}\right) \end{aligned}$$

with the ψ_k forming a bounded subset of \mathcal{S}_0 . Thus, by Proposition 3.3, T is a Calderón-Zygmund operator. \square

3.2 Two-Sided Convolution Operators

In this section, we turn to the definition of so-called “product kernels,” and use that to define the relevant “two-sided convolution operators” that we will be studying (see (1)). Our main reference for product kernels is [NRS01], and we refer the reader there for any further reading.

Definition 3.9. A product kernel on $G \times G$ is a distribution $K(x, y)$ on $G \times G$, which coincides with a C^∞ function away from $\{x = 0\} \cup \{y = 0\}$ and which satisfies:

1. (Differential inequalities) For each ordered multi-indices α_1, α_2 , there is a constant $C = C(\alpha_1, \alpha_2)$ such that:

$$\left| \nabla_{L,x}^{\alpha_1} \nabla_{L,y}^{\alpha_2} K(x, y) \right| \leq C |x|^{-Q-|\alpha_1|} |y|^{-Q-|\alpha_2|}$$

the definition remains unchanged if we replace ∇_L by ∇_R .

2. (Cancellation conditions) Given any normalized bump function ϕ on G , and any $R > 0$, the distributions:²

$$K_{\phi,R}(x) = \int K(x, y) \phi(Ry) dy$$

$$K_{\phi,R}(y) = \int K(x, y) \phi(Rx) dx$$

are Calderón-Zygmund kernels, uniformly in ϕ and R .

For such a kernel we define $\text{Op}_L(K)$ and $\text{Op}_R(K)$, acting on functions in $C_0^\infty(G \times G)$ to be convolution with K on the right and left over the group $G \times G$, respectively.

For a function $f : G \rightarrow \mathbb{C}$ and $r_1, r_2 > 0$, we define:

$$f^{(r_1, r_2)}(x, y) = r_1^Q r_2^Q f(r_1 x, r_2 y)$$

Definition 3.10. Let $\mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$ denote the set of those functions $f \in \mathcal{S}(G \times G)$ such that for every multi-index α ,

$$\int x^\alpha f(x, y) dx = 0$$

$$\int y^\alpha f(x, y) dy = 0$$

² $\int K(x, y) \phi(x) dx$ denotes that distribution which, when paired with the test function $\psi(y)$, equals $\int K(x, y) \phi(x) \psi(y) dx dy$

Remark 3.11. We note that $\mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$ is nothing more than the tensor product of the nuclear space \mathcal{S}_0 with itself. This explains our notation. See [Trè67] for a background on tensor products. We will not use any deep results about tensor products, however they will provide us with one small convenience. Since the above tensor product agrees with the projective tensor product, when we wish to prove a result about $f \in \mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$, it will often suffice to prove the result for $f(x, y) = \phi_1(x) \phi_2(y)$, $\phi_1, \phi_2 \in \mathcal{S}_0$. We will use this fact freely in the sequel. In particular, each $f(x, y) \in \mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$ can be decomposed:

$$f = \nabla_{L,x} \cdot \nabla_{R,y} \cdot g \quad (12)$$

where $g \in (\mathcal{S}_0 \hat{\otimes} \mathcal{S}_0)^{n^2}$; this can be easily seen for elementary tensor products by Lemma 3.7, and therefore holds for all elements of $\mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$.

Proposition 3.12. *Let K be a distribution on $G \times G$. Then, K is a product kernel if and only if there exists a sequence $\{\phi_{j,k}\}_{j,k \in \mathbb{Z}} \subset \mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$, forming a bounded subset of $\mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$, such that:*

$$K = \sum_{j,k \in \mathbb{Z}} \phi_{j,k}^{(2^j, 2^k)}$$

where this sum is taken in distribution (any such sum converges in distribution). In this case,

$$\text{Op}_L(K) = \sum_{j,k \in \mathbb{Z}} \text{Op}_L\left(\phi_{j,k}^{(2^j, 2^k)}\right)$$

where this sum converges in the strong operator topology as maps $L^p(G \times G) \rightarrow L^p(G \times G)$, for $(1 < p < \infty)$. In particular, $\text{Op}_L(K)$ extends to a bounded operator on $L^p(G \times G)$, $(1 < p < \infty)$. All of the above can be done uniformly for kernels forming a bounded subset of the product kernels. A similar result holds for $\text{Op}_R(K)$.

Proof. This is essentially contained in the proofs of Theorem 2.2.1, Theorem 2.6.1, and Proposition 2.7.1 of [NRS01]. We leave the details to the interested reader. \square

Given a product kernel K on $G \times G$, we may define the two-sided convolution operator $\text{Op}_T(K)$ as:

$$\text{Op}_T(K) f(x) = \int K(y, z) f(z^{-1}xy^{-1}) dy dz$$

To see that this makes sense, for x fixed, after the z integration, the integrand left over is $O(\|y\|^{-2Q})$ and so converges absolutely for y large. For y small, this makes sense using the usual pairing of distributions and test functions. Note that,

$$\text{Op}_T(K_1(x) K_2(y)) = \text{Op}_L(K_1) \text{Op}_R(K_2)$$

and so two-sided convolution operators contain the right and left invariant Calderón-Zygmund operators as special cases.

Since our algebra \mathcal{A} will contain the two-sided convolution operators corresponding to product kernels, the L^p boundedness ($1 < p < \infty$) of two-sided convolution operators will follow from the L^p boundedness of operators in \mathcal{A} . However, we will actually need the L^p boundedness of two-sided convolution operators to *prove* the L^p boundedness of operators in \mathcal{A} , and therefore we now turn to proving the L^p boundedness of two-sided convolution operators.

We now introduce a formal trick that allows us to consider two-sided convolution operators on G as convolution operators on $G \times G$. This may be found in [Kis95]. For a function $f : G \rightarrow \mathbb{C}$, we may define a new function $Ef : G \times G \rightarrow \mathbb{C}$ by

$$(Ef)(x, y) = f(y^{-1}x)$$

Then, a simple computation yields that, whenever it makes sense,

$$E\text{Op}_T(K)f = \text{Op}_L(\tilde{K})Ef$$

where $\tilde{K}(x, y) = K(x, y^{-1})$. Note that \tilde{K} is a product kernel if and only if K is. Note also that,

$$\begin{aligned} E\text{Op}_T(K_1)\text{Op}_T(K_2) &= \text{Op}_L(\tilde{K}_1)\text{Op}_L(\tilde{K}_2)E \\ &= \text{Op}_L(\tilde{K}_1 * \tilde{K}_2)E \\ &= E\text{Op}_T(\widetilde{\tilde{K}_1 * \tilde{K}_2}) \end{aligned} \tag{13}$$

and so two-sided convolution operators form an algebra (since product kernels form an algebra under convolution, see [NRS01]).

Lemma 3.13. *Suppose K is a product kernel and K has compact support. Then, $\text{Op}_T(K)$ extends to a bounded operator on $L^p(G)$, and moreover,*

$$\|\text{Op}_T(K)\|_{L^p(G) \odot} \leq \|\text{Op}_L(\tilde{K})\|_{L^p(G \times G) \odot}$$

Proof. The proof uses transference. We follow the outline of a similar argument on page 483 of [Ste93], where the proof falls under the heading “method of descent.” We suppose that $K(x, y)$ is supported on $\|x\|, \|y\| \leq M$. Consider, for $R > 0$,

$$\begin{aligned} \|\text{Op}_T(K)f\|_{L^p(G)}^p &= \|(E\text{Op}_T(K)f)(\cdot, x_2)\|_{L^p(G)}^p, \quad \forall x_2 \in G \\ &= \left\| \left(\text{Op}_L(\tilde{K})Ef \right) (\cdot, x_2) \right\|_{L^p(G)}^p, \quad \forall x_2 \in G \\ &= \frac{1}{R^Q} \int_{\|x_2\| \leq R} \left\| \left(\text{Op}_L(\tilde{K})Ef \right) (\cdot, x_2) \right\|_{L^p(G)}^p dx_2 \end{aligned} \tag{14}$$

But,

$$\left(\text{Op}_L(\tilde{K})Ef\right)(x_1, x_2) = \int \tilde{K}(y^{-1}x_1, z^{-1}x_2) Ef(y, z) dy dz$$

and since in (14), we are only considering $\|x_2\| \leq R$, and by the support of \tilde{K} only considering $\|z^{-1}x_2\| \leq M$, we see that we are only integrating over $\|z\| \leq M + R$. Hence, in (14) we may replace $Ef(y, z)$ with

$$F(y, z) := (Ef(y, z)) \chi_{\{\|z\| \leq M+R\}}$$

Thus,

$$\begin{aligned} \|\text{Op}_T(K)f\|_{L^p(G)}^p &= \frac{1}{R^Q} \int_{\|x_2\| \leq R} \left\| \left(\text{Op}_L(\tilde{K})F\right)(\cdot, x_2) \right\|_{L^p(G)}^p dx_2 \\ &\leq \frac{1}{R^Q} \left\| \text{Op}_L(\tilde{K})F \right\|_{L^p(G \times G)}^p \\ &\leq \frac{1}{R^Q} \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ}^p \|F\|_{L^p(G \times G)}^p \\ &\leq \frac{1}{R^Q} \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ}^p \int |(Ef)(y, z) \chi_{\{\|z\| \leq M+R\}}|^p dy dz \\ &= \frac{1}{R^Q} \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ}^p (M+R)^Q \|f\|_{L^p(G)}^p \\ &\xrightarrow{R \rightarrow \infty} \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ}^p \|f\|_{L^p(G)}^p \end{aligned}$$

Completing the proof. \square

Corollary 3.14. *Let K be a product kernel, then $\text{Op}_T(K)$ extends uniquely to a bounded operator on $L^p(G)$, $1 < p < \infty$, and*

$$\|\text{Op}_T(K)\|_{L^p(G) \circ} \leq \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ}$$

Proof. It suffices to show that for $f \in C_0^\infty(G)$,

$$\|\text{Op}_T(K)f\|_{L^p(G)} \leq \left\| \text{Op}_L(\tilde{K}) \right\|_{L^p(G \times G) \circ} \|f\|_{L^p(G)}$$

and this follows from Lemma 3.13 and a simple limiting argument. \square

Proposition 3.15. *Suppose K is a product kernel, and suppose that $\{\phi_{j,k}\} \subset \mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$ is a bounded subset such that*

$$K = \sum_{j,k \in \mathbb{Z}} \phi_{j,k}^{(2^j, 2^k)}$$

where this sum is taken in distribution. Then,

$$\text{Op}_T(K) = \sum_{j,k \in \mathbb{Z}} \text{Op}_T\left(\phi_{j,k}^{(2^j, 2^k)}\right)$$

where this sum converges in the strong operator topology on $L^p(G)$, $1 < p < \infty$.

Proof. Corollary 3.14 tells us that the operators:

$$\sum_{|j|,|k|\leq N} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) = \text{Op}_T \left(\sum_{|j|,|k|\leq N} \phi_{j,k}^{(2^j, 2^k)} \right)$$

are uniformly bounded on L^p . It is easy to see that for $f \in C_0^\infty$,

$$\sum_{j,k \in \mathbb{Z}} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) f = \text{Op}_T(K) f$$

where this sum is taken in distribution. Putting these two facts together, and using the fact that functions of the form $f = X_L X_R g$, $g \in C_0^\infty$, $X_L = \nabla_L^\alpha$, $|\alpha| = 1$, $X_R = \nabla_R^\beta$, $|\beta| = 1$, span a dense subset of L^p , it suffices to show that:

$$\sum_{j,k \in \mathbb{Z}} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) f \quad (15)$$

converges in L^p , for all such f . We separate (15) into four sums, and we first consider:

$$\begin{aligned} \sum_{j \geq 0, k \geq 0} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) f &= \sum_{|\alpha|=1, |\beta|=1} \sum_{j \geq 0, k \geq 0} \text{Op}_T \left(\left(\nabla_{L,x}^\alpha \nabla_{R,x}^\beta \psi_{j,k} \right)^{(2^j, 2^k)} \right) f \\ &= \sum_{\alpha, \beta} \sum_{j \geq 0, k \geq 0} 2^{-j-k} \text{Op}_T \left(\psi_{j,k}^{(2^j, 2^k)} \right) \nabla_L^\alpha \nabla_R^\beta f \end{aligned} \quad (16)$$

here we have applied (12) and $\psi_{j,k}$ also depends on α, β and ranges over a bounded subset of $\mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$ as α, β, j , and k vary. Thus, Corollary 3.14 tells us that (16) converges in L^p . We now consider:

$$\begin{aligned} \sum_{j < 0, k < 0} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) f &= \sum_{j < 0, k < 0} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right) X_L X_R g \\ &= \sum_{j < 0, k < 0} \text{Op}_T \left(X_L X_R \phi_{j,k}^{(2^j, 2^k)} \right) g \\ &= \sum_{j < 0, k < 0} 2^{j+k} \text{Op}_T \left((X_L X_R \phi_{j,k})^{(2^j, 2^k)} \right) g \end{aligned} \quad (17)$$

and since $X_L X_R \phi_{j,k}$ ranges over a bounded subset of $\mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$, we again have by Corollary 3.14 that (17) converges in L^p . Finally, the sums where $j < 0, k \geq 0$ and $k < 0, j \geq 0$ follow from a combination of the two methods above. We leave the details to the reader. \square

We state, without proof, a result similar to Theorem 3.4, that gives a characterization of two-sided convolution operators. We leave out to proof for two

reasons. Firstly, it is entirely analogous to the proof of Theorem 3.4. Secondly, we will not use this characterization for anything other than motivation for Definition 2.5.

Theorem 3.16. *Suppose $T : \mathcal{S}_0 \rightarrow C_0^\infty(G)'$. Then, $T = \text{Op}_T(K)$ (when restricted to \mathcal{S}_0), where K is a product kernel, if and only if for every $\phi \in \mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$, and every $r_L, r_R > 0$,*

$$T \text{Op}_T \left(\phi^{(r_L, r_R)} \right) = \text{Op}_T \left(\psi_{r_L, r_R}^{(r_L, r_R)} \right)$$

where $\psi_{r_L, r_R} \in \mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$, and ranges over a bounded subset of $\mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$ as ϕ ranges over a bounded subset and r_L and r_R vary.

4 Carnot-Carathéodory Distances

In this section, we review the metrics defined naturally in terms of a given family of vector fields (often called Carnot-Carathéodory metrics, or sub-Riemannian metrics). Our main references for this section are [NSW85, NS01] however we will need to restate many of the results from those papers in a slightly stronger way; though no new proofs will be required. The expert in these topics may skip this section (except, perhaps, for Section 4.1), given the understanding that all the facts we will use about such distances are true *uniformly* for $\rho_\epsilon^L, \rho_\epsilon^R$ for $\epsilon \in [0, 1]$, where $\rho_\epsilon^L, \rho_\epsilon^R$ were defined in the introduction.

Let $\Omega \subset \mathbb{R}^N$ be a connected open set, and let Y_1, Y_2, \dots, Y_q be a list (possibly with repetitions) of real C^∞ vector fields on Ω . Associate to each Y_j an integer $d_j \geq 1$, called the formal degree of Y_j . Following [NSW85, NS01], we define:

Definition 4.1 (Definition 2.1.1 in [NS01]). The list of vector fields and associated formal degrees $\{(Y_j, d_j)\}$ is said to be of finite homogeneous type on Ω if:

1. For all $1 \leq j, k \leq q$,

$$[Y_j, Y_k] = \sum_{d_l \leq d_j + d_k} c_{j,k}^l(x) Y_l$$

where $c_{j,k}^l \in C^\infty(\Omega)$.

2. At each point $x \in \Omega$, $\{Y_1(x), \dots, Y_q(x)\}$ spans the tangent space at x .

A fundamental example of Definition 4.1 (and the only one we will use) is given by a set of vector fields X_1, \dots, X_n on Ω such that all the iterated commutators of length at most m span the tangent space at each point. We take Y_1, \dots, Y_q to be a list of all these commutators, with the degree of Y_j being the length of the commutator from which it arises.

Definition 4.2 (Definition 2.1.2 of [NS01]). Let $Y = \{(Y_1, d_1), \dots, (Y_q, d_q)\}$ be a list of vector fields and formal degrees which are of finite homogeneous type on Ω . For each $\delta > 0$ let $C(\delta, Y)$ denote the set of absolutely continuous curves $\phi : [0, 1] \rightarrow \Omega$ which satisfy:

$$\phi'(t) = \sum_{j=1}^q a_j(t) Y_j(\phi(t)) \quad \text{with} \quad |a_j(t)| \leq \delta^{d_j}$$

for almost all $t \in [0, 1]$. For $x, y \in \Omega$, set

$$\rho_Y(x, y) = \inf \left\{ \delta > 0 \mid \left(\exists \phi \in C(\delta) \right) \left(\phi(0) = x, \phi(1) = y \right) \right\}$$

The function ρ_Y is called the control metric on Ω , generated by Y .

Remark 4.3. If we take Y to be that list of vector fields generated by the left invariant vector fields of order 1 (ie, we take ∇_L^α , $|\alpha| = 1$ to be the vector fields whose iterated commutators up to some order span the tangent space, and use these to generate Y as discussed above), then the induced metric $\rho_Y(x, y)$ is equivalent to $\|y^{-1}x\|$. Indeed, it is easy to see that they are both left invariant, and both homogeneous of order 1 with respect to the dilations on the group, and the equivalence then follows from a simple compactness argument.

If $Y = \{(Y_1, d_1), \dots, (Y_q, d_q)\}$ is of finite homogeneous type, and if $I = (i_1, \dots, i_N)$ is an ordered N -tuple of integers, with each $i_j \leq q$, we define:

$$\lambda_I^Y(x) = \det(Y_{i_1}, \dots, Y_{i_N})(x)$$

where we regard each Y_i as an N -tuple of smooth functions, and λ_I^Y is then the determinant of the corresponding $N \times N$ matrix. We also set:

$$d(I) = d_{i_1} + \dots + d_{i_N}$$

and define:

$$\Lambda_Y(x, \delta) = \sum_I |\lambda_I^Y(x)| \delta^{d(I)}$$

Definition 4.4. Let S be a set of lists of vector fields $Y = \{(Y_1, d_1), \dots, (Y_q, d_q)\}$ of homogeneous type, where q may vary with Y . We say S is bounded if there is a uniform bound for q for all $Y \in S$, and the following hold:

1. There is an M such that $d_j \leq M$ for all formal degrees associated to some $Y \in S$.
2. We insist that the set of vector fields listed in some $Y \in S$, thought of as sections of $T\Omega$, form a bounded set in the usual topology of smooth sections of $T\Omega$.
3. The $c_{j,k}^l$ from Definition 4.1 may be chosen from a bounded subset of C^∞ uniformly for $Y \in S$.

4. For every compact set $K \subset \Omega$, there exists a $c > 0$ such that for each $Y \in S$ we have $\Lambda_Y(x, 1) \geq c$ for $x \in K$.

The relevance of such bounded sets S is that many of the results in [NSW85, NS01] hold uniformly for $Y \in S$, with no changes to the proof. We shall need some of these results and state them below. We remark that these bounded sets are the precompact sets in a natural topology on the set of families of vector fields of homogeneous type; though we do not expound on this further, as it will be of no use to us in the sequel. Fix, for the remainder of this section, such a bounded set S . We will remind the reader of the results from [NSW85, NS01] that we will use, and make explicit their uniformity in S . All of the results in this section follow by merely keeping track of the constants' dependence on Y in [NSW85, NS01].

Remark 4.5. The reader wishing to prove the results in this section may find it useful to recall that the inverse function theorem remains true uniformly for compact subsets of C^∞ . Ie, if $R \subset C^\infty$ is a compact set, and if x is a point such that for all $f \in R$, the Jacobian determinant of f at x is non-zero (and hence, has absolute value bounded below, independent of f), then there exists an open neighborhood U (independent of f) containing x such that for all $f \in R$, $f : U \rightarrow f(U)$ is a diffeomorphism. The essential point here is actually that R is a compact subset of C^1 .

For a list of vector fields and formal degrees $Y = \{(Y_1, d_1), \dots, (Y_q, d_q)\}$ of homogeneous type, we define:

$$B_Y(x, \delta) = \{y : \rho_Y(x, y) < \delta\}$$

Definition 4.6. We say that two functions $\rho_1, \rho_2 : \Omega \times \Omega \rightarrow [0, \infty]$ are locally equivalent if for every $x_0 \in \Omega$ there exists an open set U containing x_0 such that for every compact set $K \subset\subset U$ there is a constant C such that if $x_1, x_2 \in K$,

$$\frac{1}{C} \rho_1(x_1, x_2) \leq \rho_2(x_1, x_2) \leq C \rho_1(x_1, x_2)$$

[NSW85] defines other pseudo-distances that are locally equivalent to ρ_Y , but can be easier to work with. We remind the reader of two of them that we shall use. The definition of the first is similar to that of ρ_Y , but only allows constant linear combinations of the vectors Y_1, \dots, Y_q . For $\delta > 0$ let $C_2(\delta, Y)$ denote the class of smooth curves $\phi : [0, 1] \rightarrow \Omega$ such that:

$$\phi'(t) = \sum_{j=1}^q a_j Y_j(\phi(t))$$

with $|a_j| < \delta^{d_j}$. Define:

$$\rho_{Y,2}(x, y) = \inf \left\{ \delta > 0 : \left(\exists \phi \in C_2(\delta, Y) \right) \left(\phi(0) = x, \phi(1) = y \right) \right\}$$

Theorem 4.7 (Theorem 2 from [NSW85]). $\rho_{2,Y}$ is locally equivalent to ρ_Y , with constants that can be chosen uniformly for $Y \in S$.

The definition of the second locally equivalent metric allows us to single out N of the vector fields Y_{i_1}, \dots, Y_{i_N} . For each N -tuple $I = (i_1, \dots, i_N)$, let $C_3(\delta, I, Y)$ denote the class of smooth curves $\phi : [0, 1] \rightarrow \Omega$ such that:

$$\phi'(t) = \sum_{j=1}^N a_j Y_{i_j}(\phi(t))$$

with $|a_j| < \delta^{d(Y_{i_j})}$. We define

$$\rho_{Y,3}(x, y) = \inf \left\{ \delta > 0 : \left(\exists I \exists \phi \in C_3(\delta, I, Y) \right) \left(\phi(0) = x, \phi(1) = y \right) \right\}$$

Theorem 4.8 (Theorem 3 from [NSW85]). $\rho_{3,Y}$ is locally equivalent to ρ_Y , with constants that can be chosen uniformly for $Y \in S$. Moreover, for every $x_0 \in \Omega$, there exists an open set U containing x_0 such that for every compact set $K \subset\subset U$ we have the following for all $Y \in S$: if for a fixed $x \in K$ we have:

$$\delta^{d(I)} |\lambda_I^Y(x)| \geq \epsilon \Lambda_Y(x, \delta)$$

then, there exists a C depending on ϵ and K , but not on Y , such that for every $y \in K$,

$$\rho_{3,Y}(x, y) \geq C \inf \left\{ \delta > 0 : \left(\exists \phi \in C_3(\delta, I, Y) \right) \left(\phi(0) = x, \phi(1) = y \right) \right\}$$

Theorem 4.9 (Theorem 1 from [NSW85]). For every compact set $K \subset\subset \Omega$, there are constants C_1, C_2 such that for all $x \in K$ and all $Y \in S$,

$$0 < C_1 \leq \frac{|B_Y(x, \delta)|}{\Lambda_Y(x, \delta)} \leq C_2$$

where here, and in the rest of the paper, $|E|$ denotes the Lebesgue measure of E .

Corollary 4.10 (Corollary of Theorem 4.9). For every compact set $K \subset\subset \Omega$ there is a constant C such that for all $Y \in S$ and all $x \in K$,

$$|B(x, 2\delta)| \leq C |B(x, \delta)|$$

Theorem 4.11 (Lemma 3.1.1 from [NS01]). Let $E \subset\subset \Omega$ be compact. There exist constants $\delta_0, \epsilon_0, \sigma_0 > 0$ such that for each $x \in E$, each $0 < \delta < \delta_0$, and each $Y \in S$, there is a function $\phi = \phi_{x, \delta, Y} \in C^\infty(\Omega)$ such that:

1. For all $y \in \Omega$, $0 \leq \phi(y) \leq 1$.
2. $\phi(y) = 0$ when $\rho(x, y) > \sigma_0 \delta$, $\phi(y) = 1$ when $\rho(x, y) < \epsilon_0 \delta$.

3. For every ordered multi-index α ,

$$\sup_{y \in \Omega} |\nabla_Y^\alpha \phi(y)| \leq C_\alpha \delta^{-d(\alpha)}$$

where $\nabla_Y = (Y_1, \dots, Y_q)$, and $d(\alpha)$ is the formal degree of ∇_Y^α with each Y_j having formal degree d_j .

Remark 4.12. Actually, the stronger results given by Theorem 3.3.1 and Theorem 3.3.2 of [NS01] are true uniformly for $Y \in S$, but we will not need these results.

4.1 A New Distance

Given two metrics, ρ_1 and ρ_2 , one may define a third function $\rho_2 \circ \rho_1 : \Omega \times \Omega \rightarrow [0, \infty]$ defined by:

$$\rho_2 \circ \rho_1(x, y) = \inf \{ \delta > 0 : \exists z \in \Omega, \rho_1(x, z) \leq \delta, \rho_2(z, y) \leq \delta \}$$

Suppose $X = \{(X_1, c_1), \dots, (X_p, c_p)\}$ and $Y = \{(Y_1, d_1), \dots, (Y_q, d_q)\}$ are two list of vector fields that are of finite homogeneous type. Suppose also that $[X_j, Y_k] = 0$ for every $1 \leq j \leq p, 1 \leq k \leq q$.

We will see (under an assumption) that

$$\rho_Y \circ \rho_X = \rho_{X \cup Y}$$

In fact, even without our assumption, our proof works locally. However, the condition that the X_j commute with the Y_k is so restrictive this is a moot point (see Remark 4.15).

Before we speak about our assumption, a word of notation. If (Z, d) appears in both X and Y we count it as appearing twice in $X \cup Y$, equivalently, we replace (Z, d) with $(2Z, d)$ in $X \cup Y$.

Our assumption is as follows: for every $\delta > 0$ and every $|a_j(t)| \leq \delta, 1 \leq j \leq p$, a_j measurable, and every $x \in \Omega$, there exists a unique solution in Ω to:

$$\phi(0) = x$$

$$\phi'(t) = \sum_{j=1}^p a_j(t) X_j(\phi(t))$$

and similarly for the Y_k . We denote this solution by the time-ordered exponential (also known as the product integral):

$$\phi(t) = \text{T-exp} \left(\int_0^t \sum_{j=1}^p a_j(s) X_j ds \right) x$$

See [DF79, GAV89] for a background on product integration.

If $Z(s)$ is a family of vector fields (and is locally integrable), then:

$$\text{T-exp} \left(\int_0^t Z(s) ds \right) x = \lim_{N \rightarrow \infty} \exp \left(\int_{\frac{N-1}{N}t}^t Z(s) ds \right) \cdots \exp \left(\int_0^{\frac{t}{N}} Z(s) ds \right) x$$

From here we see that if $Z_1(s)$ and $Z_2(r)$ commute for every s and r , then:

$$\text{T-exp} \left(\int_0^t Z_1(s) + Z_2(s) ds \right) x = \text{T-exp} \left(\int_0^t Z_1(s) ds \right) \text{T-exp} \left(\int_0^t Z_2(s) ds \right) x \quad (18)$$

Theorem 4.13. *Under the setup above, we have:*

$$\rho_Y \circ \rho_X = \rho_{X \cup Y}$$

Proof. Suppose that $\rho_Y \circ \rho_X(x, y) < \delta$, so that there exists a z with $\rho_X(x, z) < \delta$ and $\rho_Y(z, y) < \delta$. Let $\phi_X, \phi_Y : [0, 1] \rightarrow \Omega$ be absolutely continuous curves such that $\phi_X(0) = x$, $\phi_X(1) = z$, $\phi_Y(0) = z$, and $\phi_Y(1) = y$, and such that

$$\begin{aligned} \phi_X(t) &= \text{T-exp} \left(\int_0^t \sum_{j=1}^p a_j(s) X_j ds \right) x \\ \phi_Y(t) &= \text{T-exp} \left(\int_0^t \sum_{k=1}^q b_k(s) Y_k ds \right) z \end{aligned}$$

with $|a_j| < \delta^{c_j}$, $|b_j| < \delta^{d_j}$. But then,

$$\gamma(t) = \text{T-exp} \left(\int_0^t \sum_{j=1}^p a_j(s) X_j + \sum_{k=1}^q b_k(s) Y_k ds \right) x$$

is a path from x to z . Indeed,

$$\begin{aligned} \gamma(1) &= \text{T-exp} \left(\int_0^1 \sum_{j=1}^p a_j(s) X_j + \sum_{k=1}^q b_k(s) Y_k ds \right) x \\ &= \text{T-exp} \left(\int_0^1 \sum_{k=1}^q b_k(s) Y_k ds \right) \text{T-exp} \left(\int_0^1 \sum_{j=1}^p a_j(s) X_j ds \right) x \\ &= \text{T-exp} \left(\int_0^1 \sum_{k=1}^q b_k(s) Y_k ds \right) z \\ &= y \end{aligned}$$

But, we also have that:

$$\gamma'(t) = \sum_{j=1}^p a_j(t) X_j(\gamma(t)) + \sum_{k=1}^q b_k(t) Y_k(\gamma(t))$$

and since $|a_j| < \delta^{c_j}$ and $|b_j| < \delta^{d_j}$ we see that $\rho_{X \cup Y}(x, y) < \delta$.

Conversely, suppose $\rho_{X \cup Y}(x, y) < \delta$. Then there is a path of the form:

$$\phi(t) = \text{T-exp} \left(\int_0^t \sum_{j=1}^p a_j(s) X_j + \sum_{k=1}^q b_k(s) Y_k ds \right) x$$

with $\phi(1) = y$, $|a_j| < \delta^{c_j}$, $|b_k| < \delta^{d_k}$. Define

$$\phi_X(t) = \text{T-exp} \left(\int_0^t \sum_{j=1}^p a_j(s) X_j ds \right) x$$

and let $z = \phi_X(1)$. Define:

$$\phi_Y(t) = \text{T-exp} \left(\int_0^t \sum_{k=1}^q b_k(s) Y_k ds \right) z$$

Note that $\phi_Y(1) = \phi(1) = y$ by (18). It is easy to see that $\phi_X \in C(\delta, X)$, $\phi_Y \in C(\delta, Y)$, and it then follows that $\rho_X(x, z) < \delta$ and $\rho_Y(z, y) < \delta$, showing that $\rho_Y \circ \rho_X(x, y) < \delta$ and completing the proof. \square

Remark 4.14. If one wished to show only that $\rho_Y \circ \rho_X$ is locally equivalent to $\rho_{X \cup Y}$ in Theorem 4.13 (which would be sufficient for our purposes), then the proof is a bit easier. Indeed, Theorem 4.7 would allow us to replace the exponentials with variable coefficients with ones with constant coefficients. Then the same proof yields the result, without any need for time ordered exponentials, nor the need for our assumption. In spite of this, we believe that the proof of Theorem 4.13 helps to elucidate the situation.

Remark 4.15. One example of such X_j and Y_k is as follows: let X_j be a spanning set of the right invariant vector fields on some Lie group, and let Y_k be a spanning set of the left invariant vector fields (and we may even restrict them to a small connected open set). It is not hard to see that this is the only example.

5 The Distances ρ_ϵ^L and ρ_ϵ^R

Given a finite set F of vector fields such that F along with the commutators of all orders of elements of F up to some fixed order m :

$$[X_1, [X_2, \dots, [X_{n-1}, X_n] \dots]], \quad n \leq m, \quad X_j \in F$$

span the tangent space at each point (it is often said that such a set F satisfies Hörmander's condition), we associate a list of vector fields of finite homogeneous type as in Section 4, by taking the list of all commutators up to order m and associating to a commutator of length k degree k . That is to say the elements of F are given degree 1, elements of the form $[X_1, X_2]$ are given degree 2 and so

forth. Call this list of vector fields $\mathcal{L}(F)$. From this list of vector fields of finite homogeneous type, we get a metric $\rho_{\mathcal{L}(F)}$.

We define, as in the introduction for $\epsilon \in [0, 1]$,

$$\rho_\epsilon^L = \rho_{\mathcal{L}(\{\nabla_L, \epsilon \nabla_R\})}$$

$$\rho_\epsilon^R = \rho_{\mathcal{L}(\{\nabla_R, \epsilon \nabla_L\})}$$

Here, our set Ω from Section 4 is the entire group G . Now it is easy to see that

$$\{\mathcal{L}(\{\nabla_L, \epsilon \nabla_R\}), \mathcal{L}(\{\nabla_R, \epsilon \nabla_L\}) \mid \epsilon \in [0, 1]\}$$

is a bounded set as in Section 4. Thus, all of the theorems from that section hold uniformly for $\epsilon \in [0, 1]$.

We define $B_\epsilon^L(x, \delta)$ to be the ball of radius δ centered at x in the ρ_ϵ^L metric, and we define $V_\epsilon^L(x, \delta)$ to be its volume. Similarly, we define $B_\epsilon^R(x, \delta)$ and $V_\epsilon^R(x, \delta)$. Note that all of the relevant quantities from Section 4 are homogeneous of an appropriate degree. For instance,

$$\rho_\epsilon^L(rx, ry) = r \rho_\epsilon^L(x, y)$$

$$V_\epsilon^L(rx, \delta) = r^Q V_\epsilon^L\left(x, \frac{\delta}{r}\right)$$

From such considerations, it is easy to see that all of the results from Section 4 hold globally, instead of locally. That is, many of the results are true on any fixed compact set E . Take that compact set E to be the closed unit ball. Then to see that the result holds globally, merely scale everything down until it fits into the unit ball, and apply the result on the unit ball. As all the quantities are homogeneous of the proper degrees, this extends the results. In the same manner, we may even take $\delta_0 = \infty$ in Theorem 4.11.

Remark 5.1. We have the following scaling properties of the distances ρ_ϵ^L and ρ_ϵ^R :

$$r \rho_\epsilon^L(x, y) = \rho_\epsilon^L(rx, ry) = \rho_{\{\frac{1}{r} \nabla_L, \frac{1}{r} \epsilon \nabla_R\}}(x, y)$$

and similarly for ρ_ϵ^R . The first equality just follows by homogeneity of the vector fields and was discussed above. To see that the first term equals the last term, note that:

$$C(\delta, \mathcal{L}(\nabla_L, \epsilon \nabla_R)) = C\left(r\delta, \mathcal{L}\left(\frac{1}{r} \nabla_L, \frac{1}{r} \epsilon \nabla_R\right)\right)$$

as can be seen directly from the definition.

5.1 Relationship to Convolution Operators

In this section, we investigate the relationship between ρ_ϵ^L , ρ_ϵ^R and two-sided convolution operators on G . We will use one simplifying piece of notation. For an operator T , we write $\text{Ker}(T)(x, y)$ for the Schwartz kernel of T when mapping from the y variable to the x variable.

Define $B := \{x : \|x\| < 1\}$. With χ_B denoting the characteristic function of B , set (for $r_L, r_R > 0$):

$$\begin{aligned} K_{r_L, r_R}(x, y) &= \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} \right) \right) (x, y) \\ &= r_L^Q r_R^Q \int \chi_{\frac{1}{r_L} B}(z^{-1}x) \chi_{\frac{1}{r_R} B}(zy^{-1}) dz \\ &= \left| \left\{ z : \|z^{-1}x\| < \frac{1}{r_L}, \|zy^{-1}\| < \frac{1}{r_R} \right\} \right| \end{aligned} \quad (19)$$

Recall that $\|z^{-1}x\| \approx \rho_0^L(x, z)$ (see Remark 4.3) and, similarly, $\|zy^{-1}\| \approx \rho_0^R(y, z)$. Using that $r_L \rho_0^L(x, y) = \rho_{\mathcal{L}}(\frac{1}{r_L} \nabla_L)$ (and similarly for $r_R \rho_0^R$) (see Remark 5.1), and using (19), we see that there exists a constant C (independent of r_L, r_R) such that:

$$\begin{aligned} \left\{ (x, y) : \rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R\right) \circ \rho_{\mathcal{L}}\left(\frac{1}{r_L} \nabla_L\right) (x, y) \leq C^{-1} \right\} &\subseteq \{(x, y) : K_{r_L, r_R}(x, y) \neq 0\} \\ &\subseteq \left\{ (x, y) : \rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R\right) \circ \rho_{\mathcal{L}}\left(\frac{1}{r_L} \nabla_L\right) (x, y) \leq C \right\} \end{aligned} \quad (20)$$

Section 4.1 tells us that:

$$\rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R\right) \circ \rho_{\mathcal{L}}\left(\frac{1}{r_L} \nabla_L\right) = \rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R\right) \cup \rho_{\mathcal{L}}\left(\frac{1}{r_L} \nabla_L\right) = \rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R, \frac{1}{r_L} \nabla_L\right)$$

while Remark 5.1 tells us that (when $r_L \leq r_R$):

$$\rho_{\mathcal{L}}\left(\frac{1}{r_R} \nabla_R, \frac{1}{r_L} \nabla_L\right) = r_L \rho_{\mathcal{L}}\left(\frac{r_L}{r_R} \nabla_R, \nabla_L\right) = r_L \rho_{\frac{r_L}{r_R}}^L$$

with a similar result with $r_R \leq r_L$. For the remainder of this section, we restrict our attention to the case $r_L \leq r_R$, with the understanding that the case $r_R \leq r_L$ follows in the same way with completely symmetric arguments.

Putting all of this together, we see that:

$$B_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{Cr_L} \right) \subseteq \{y : K_{r_L, r_R}(x, y) \neq 0\} \subseteq B_{\frac{r_L}{r_R}}^L \left(x, \frac{C}{r_L} \right) \quad (21)$$

Applying Corollary 4.10, we see that:

$$|\{y : K_{r_L, r_R}(x, y) \neq 0\}| \approx V_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{r_L} \right)$$

Define:

$$M(x, r_L, r_R) = \sup_y K_{r_L, r_R}(x, y)$$

The main result of this section is the following theorem and its corollary:

Theorem 5.2. $M(x, r_L, r_R) \approx \frac{1}{V_{\frac{r_L}{r_R}}^L(x, \frac{1}{r_L})}$. Moreover, there is a $\delta_0 > 0$ (independent of r_L, r_R) such that for all x , there exists a y_0 with:

$$K_{r_L, r_R}(x, y) \approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$$

for all $y \in B_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right)$.

Corollary 5.3. We may take $y_0 = x$ in Theorem 5.2.

Remark 5.4. One of our main uses for Corollary 5.3 is as follows. Take δ_0 as in the corollary, and set $\chi = \chi_{\frac{1}{\delta_0}B}$. Then, it is easy to see that

$$\phi_0(y) := \text{Ker}\left(\text{Op}_L\left(\chi^{(r_L)}\right)\text{Op}_R\left(\chi^{(r_R)}\right)\right)(x, y) \gtrsim \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$$

for $y \in B_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)$. Thus, when we wish bound a function ϕ supported on $B_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)$ and which is $\lesssim \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$, it suffices to instead bound ϕ_0 .

Lemma 5.5. $M(x, r_L, r_R) \approx M\left(x, \frac{r_L}{2}, \frac{r_R}{2}\right)$

Proof. It is clear that

$$M(x, r_L, r_R) \leq 2^{2Q} M\left(x, \frac{r_L}{2}, \frac{r_R}{2}\right)$$

and so we focus only on the reverse inequality.

For the proof of this lemma, alone, we drop the assumption that $r_L \leq r_R$. Then, it suffices to show that:

$$M\left(x, r_L, \frac{r_R}{2}\right) \lesssim M(x, r_L, r_R)$$

and the remainder of the result will follow by symmetry.

Define $g = \chi_B * \chi_{2B}$. Note that there exists a $c > 0$ such that $g(x) > c$ for

$x \in 2B$. Hence, we have:

$$\begin{aligned}
K_{r_L, \frac{r_R}{2}}(x, y) &= 2^{-Q} \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_{2B}^{(r_R)} \right) \right) (x, y) \\
&\lesssim \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(g^{(r_R)} \right) \right) (x, y) \\
&= \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} * \chi_{2B}^{(r_R)} \right) \right) (x, y) \\
&= \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} \right) \text{Op}_R \left(\chi_{2B}^{(r_R)} \right) \right) (x, y) \\
&= \int K_{r_L, r_R}(x, z) \chi_{2B}^{(r_R)}(y^{-1}z) dz \\
&\leq \int M(x, r_L, r_R) \chi_{2B}^{(r_R)}(y^{-1}z) dz \\
&= M(x, r_L, r_R) \int \chi_{2B}(z) dz \\
&\approx M(x, r_L, r_R)
\end{aligned}$$

Completing the proof. \square

Let $\phi \in C^\infty(G)$ be ≥ 0 , ≤ 1 , supported on $2B$, and $= 1$ on B . Define:

$$\begin{aligned}
\tilde{K}_{r_L, r_R} &= \text{Ker} \left(\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\phi^{(r_R)} \right) \right) \\
\tilde{M}(x, r_L, r_R) &= \sup_y \tilde{K}_{r_L, r_R}(x, y)
\end{aligned}$$

So that the definition of K shows:

$$K_{r_L, r_R}(x, y) \leq \tilde{K}_{r_L, r_R}(x, y) \leq 2^{2Q} K_{\frac{r_L}{2}, \frac{r_R}{2}}(x, y)$$

and Lemma 5.5 then tells us:

$$M(x, r_L, r_R) \approx \tilde{M}(x, r_L, r_R)$$

Lemma 5.6. *There exists $C_0 > 0$ (independent of r_L, r_R) such that for any $1 \geq \delta > 0$, and any $\gamma \in C \left(\frac{\delta}{r_L}, \mathcal{L} \left(\nabla_L, \frac{r_L}{r_R} \nabla_R \right) \right)$, we have:*

$$\left| \tilde{K}_{r_L, r_R}(x, \gamma(1)) - \tilde{K}_{r_L, r_R}(x, \gamma(0)) \right| \leq C_0 \delta \tilde{M}(x, r_L, r_R)$$

Before we prove Lemma 5.6, let's see how it finishes the proof of Theorem 5.2. Fix x and let y_0 be such that:

$$\tilde{K}_{r_L, r_R}(x, y_0) = \tilde{M}(x, r_L, r_R)$$

Then if we take $\delta_0 = \delta = \min \left\{ \frac{1}{2C_0}, 1 \right\}$ in Lemma 5.6, we see that for $y \in B_{\frac{r_L}{r_R}}^L(y_0, \delta_0)$,

$$\tilde{K}_{r_L, r_R}(x, y) \geq \frac{1}{2} \tilde{M}(x, r_L, r_R)$$

Suppose, for contradiction that $\widetilde{M}(x, r_L, r_R) >> \frac{1}{V_{\frac{r_L}{r_R}}^L(x, \frac{1}{r_L})}$ (here $>>$ just means not \lesssim). Then,

$$\begin{aligned}
\int \widetilde{K}_{r_L, r_R}(x, y) dy &\geq \int_{y \in B_{\frac{r_L}{r_R}}^L(y_0, \frac{\delta_0}{r_L})} \widetilde{K}_{r_L, r_R}(x, y) dy \\
&\geq \frac{1}{2} \int_{y \in B_{\frac{r_L}{r_R}}^L(y_0, \frac{\delta_0}{r_L})} \widetilde{M}(x, r_L, r_R) dy \\
&= \frac{1}{2} \widetilde{M}(x, r_L, r_R) V_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right) \\
&>> \frac{1}{V_{\frac{r_L}{r_R}}^L(x, \frac{1}{r_L})} V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right) \\
&= 1
\end{aligned}$$

Here we used that

$$V_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right) \approx V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right) \quad (22)$$

as can be seen by the fact that since y_0 is in the support of $\widetilde{K}_{r_L, r_R}(x, y)$ we must have $\rho_{\frac{r_L}{r_R}}^L(x, y_0) \lesssim \frac{1}{r_L}$, and thus,

$$\begin{aligned}
B_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right) &\subset B_{\frac{r_L}{r_R}}^L\left(y_0, C_1 \frac{\delta_0}{r_L}\right) \\
B_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right) &\subset B_{\frac{r_L}{r_R}}^L\left(x, \frac{C_1}{r_L}\right)
\end{aligned}$$

for some C_1 .

But we also have,

$$\int \widetilde{K}_{r_L, r_R}(x, y) dy = \int \phi(x) dx \int \phi(y) dy \approx 1$$

achieving the contradiction.

Hence we see that $\widetilde{M}(x, r_L, r_R) \lesssim \frac{1}{V_{\frac{r_L}{r_R}}^L(x, \frac{1}{r_L})}$. But, $\widetilde{M}(x, r_L, r_R) \approx M(x, r_L, r_R)$,

and applying (20) we see:

$$\begin{aligned}
M(x, r_L, r_R) &\geq \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{C}{r_L}\right)} \int_{y \in B_{\frac{r_L}{r_R}}^L\left(x, \frac{C}{r_L}\right)} K_{r_L, r_R}(x, y) dy \\
&= \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{C}{r_L}\right)} \int K_{r_L, r_R}(x, y) dy \\
&= \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{C}{r_L}\right)} \\
&\approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}
\end{aligned}$$

And so we see that:

$$M(x, r_L, r_R) \approx \widetilde{M}(x, r_L, r_R) \approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$$

proving the first part of Theorem 5.2. Moreover, we therefore have for all $y \in B_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right)$,

$$\widetilde{K}_{r_L, r_R}(x, y) \approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)}$$

and since:

$$\widetilde{K}_{r_L, r_R}(x, y) \lesssim K_{\frac{r_L}{2}, \frac{r_R}{2}}(x, y)$$

we have for all $y \in B_{\frac{r_L}{r_R}}^L\left(y_0, \frac{\delta_0}{r_L}\right)$,

$$K_{\frac{r_L}{2}, \frac{r_R}{2}}(x, y) \approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L}\right)} \approx \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{2}{r_L}\right)}$$

Dividing δ_0 by 2, this proves the second part of Theorem 5.2 for $K_{\frac{r_L}{2}, \frac{r_R}{2}}$. Now merely replace r_L and r_R with $2r_L$ and $2r_R$ to complete the proof.

Proof of Lemma 5.6. We again use time-ordered exponentials, as in Section 4.1; and we again remark that their use is unnecessary, given Theorem 4.7 (see Remark 4.14), however we shall use them as we believe it adds to the clarity of the exposition.

Let $\left\{ \left(Y_j^{(L)}, d_j \right) \right\} = \mathcal{L}(\nabla_L)$ and let $\left\{ \left(Y_j^{(R)}, d_j \right) \right\} = \mathcal{L}(\nabla_R)$ so that

$$\left\{ \left(Y_j^{(L)}, d_j \right), \left(\epsilon^{d_j} Y_j^{(R)}, d_j \right) \right\} = \mathcal{L}(\nabla_L, \epsilon \nabla_R)$$

We remark that $Y_j^{(L)} f(rx) = r^{d_j} \left(Y_j^{(L)} f \right)(rx)$, and similarly for $Y_j^{(R)}$. We also remark that, if we do our enumeration consistently between right and left, we

have $Y_j^{(R)} f(x^{-1}) = (-1)^{d_j} (Y_j^{(L)} f)(x^{-1})$. Suppose $\gamma \in C\left(\frac{\delta}{r_L}, \mathcal{L}\left(\nabla_L, \frac{r_L}{r_R} \nabla_R\right)\right)$ with $\gamma(0) = z_0, \gamma(1) = z_1$; so that

$$\gamma(t) = \text{T-exp} \left(\int_0^t \sum_j a_j(s) Y_j^{(L)} + \left(\frac{r_L}{r_R}\right)^{d_j} b_j(s) Y_j^{(R)} ds \right) z_0$$

with $|b_j|, |a_j| < \left(\frac{\delta}{r_L}\right)^{d_j}$.

As in the proof of Theorem 4.13, we define:

$$\gamma_L(t) = \text{T-exp} \left(\int_0^t \sum_j a_j(s) Y_j^{(L)} ds \right) z_0$$

$$\gamma_R(t) = \text{T-exp} \left(\int_0^t \sum_j \left(\frac{r_L}{r_R}\right)^{d_j} b_j(s) Y_j^{(R)} ds \right) \gamma_L(1)$$

so that $\gamma_R(1) = z_1$. Consider,

$$\begin{aligned} \frac{d}{dt} \tilde{K}_{r_L, r_R}(x, \gamma_R(t)) &= r_L^Q r_R^Q \frac{d}{dt} \int \phi(r_L(y^{-1}x)) \phi(r_R(y\gamma_R(t)^{-1})) dy \\ &= r_L^Q r_R^Q \int \phi(r_L(y^{-1}x)) \left(\sum_j \left(-\frac{r_L}{r_R}\right)^{d_j} r_R^{d_j} b_j(t) (Y_j^{(L)} \phi)(r_R(y\gamma_R(t)^{-1})) \right) dy \end{aligned}$$

Hence, using that $|b_j| < \left(\frac{\delta}{r_L}\right)^{d_j}$, and using that ϕ is a fixed C^∞ function supported on $2B$, we see that:

$$\begin{aligned} \left| \frac{d}{dt} \tilde{K}_{r_L, r_R}(x, \gamma_R(t)) \right| &\lesssim r_L^Q r_R^Q \int \chi_{2B}(r_L(y^{-1})) \delta \chi_{2B}(r_R(y\gamma_R(t)^{-1})) dy \\ &\lesssim \delta M\left(x, \frac{r_L}{2}, \frac{r_R}{2}\right) \\ &\approx \delta M(x, r_L, r_R) \end{aligned}$$

Now, consider:

$$\begin{aligned} \tilde{K}_{r_L, r_R}(x, \gamma_L(t)) &= \int \phi^{(r_L)}(y^{-1}x) \phi^{(r_R)}(y\gamma_L(t)^{-1}) dy \\ &= \int \phi^{(r_L)}(\gamma_L(t)^{-1}y) \phi^{(r_R)}(xy^{-1}) dy \end{aligned}$$

and then a similar proof to the one above shows that:

$$\left| \frac{d}{dt} \tilde{K}_{r_L, r_R}(x, \gamma_L(t)) \right| \lesssim \delta M(x, r_L, r_R)$$

Completing the proof. \square

Proof of Corollary 5.3. It is easy to see that

$$\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} \right)$$

is a self-adjoint operator, and thus,

$$K_{r_L, r_R}(x, y) = K_{r_L, r_R}(y, x)$$

If we take y_0 as in Theorem 5.2, then we see:

$$\begin{aligned} & \text{Ker} \left(\text{Op}_L \left((\chi_B * \chi_B)^{(r_L)} \right) \text{Op}_R \left((\chi_B * \chi_B)^{(r_R)} \right) \right) (x, x) \\ &= \text{Ker} \left(\left[\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} \right) \right]^2 \right) (x, x) \\ &= \int K_{r_L, r_R}(x, y) K_{r_L, r_R}(x, y) dy \\ &\gtrsim \int_{y \in B_{\frac{r_L}{r_R}}^L(y_0, \frac{\delta_0}{r_L})} \left(\frac{1}{V_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{r_L} \right)} \right)^2 dy \\ &\approx \int_{y \in B_{\frac{r_L}{r_R}}^L(y_0, \frac{\delta_0}{r_L})} \frac{1}{V_{\frac{r_L}{r_R}}^L \left(y_0, \frac{\delta_0}{r_L} \right)} \frac{1}{V_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{r_L} \right)} dy \\ &= \frac{1}{V_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{r_L} \right)} \end{aligned}$$

where we have applied (22) to get the second to last line.

Since $\chi_B * \chi_B \leq \chi_{4B}$, we have shown:

$$\begin{aligned} K_{\frac{r_L}{4}, \frac{r_R}{4}}(x, x) &\gtrsim \frac{1}{V_{\frac{r_L}{r_R}}^L \left(x, \frac{1}{r_L} \right)} \\ &\approx \frac{1}{V_{\frac{r_L}{r_R}}^L \left(x, \frac{4}{r_L} \right)} \\ &\approx M \left(x, \frac{r_L}{4}, \frac{r_R}{4} \right) \end{aligned}$$

Replacing r_L, r_R with $4r_L, 4r_R$, we see that:

$$\widetilde{K}_{r_L, r_R}(x, x) \geq K_{r_L, r_R}(x, x) \approx M(x, r_L, r_R) \approx \widetilde{M}(x, r_L, r_R)$$

Lemma 5.6 then tells us that there is a $\delta_0 > 0$ such that for $y \in B_{\frac{r_L}{r_R}}^L(x, \delta_0)$,

$$\begin{aligned} K_{\frac{r_L}{2}, \frac{r_R}{2}}(x, y) &\geq \widetilde{K}_{r_L, r_R}(x, y) \\ &\gtrsim \widetilde{M}(x, r_L, r_R) \\ &\approx M \left(x, \frac{r_L}{2}, \frac{r_R}{2} \right) \end{aligned}$$

proving the result for $\frac{r_L}{2}, \frac{r_R}{2}$ in place of r_L, r_R . \square

We close this section with some simple inequalities that will be of use in the sequel.

Lemma 5.7. *For $r_L, r_R, r_L^1, r_R^1 > 0$,*

$$K_{r_L^1, r_R^1}(x, z) \leq \begin{cases} \left(\frac{r_L r_R^1}{r_L^1 r_R} \right)^Q K_{r_L^1, \frac{r_L^1 r_R}{r_L}}(x, z) & \text{if } \frac{r_L}{r_R} \geq \frac{r_L^1}{r_R^1} \\ \left(\frac{r_R r_L^1}{r_L^1 r_R} \right)^Q K_{\frac{r_R^1 r_L}{r_R}, r_R^1}(x, z) & \text{if } \frac{r_L}{r_R} \leq \frac{r_L^1}{r_R^1} \end{cases}$$

Proof. This follows directly from the definition. \square

Corollary 5.8. *Suppose $r_L \leq r_R$ and $r_L^1 \leq r_R^1$. Then, we have:*

$$\frac{1}{V_x^L\left(\frac{r_L^1}{r_R^1}, \delta\right)} \lesssim \begin{cases} \left(\frac{r_L r_R^1}{r_L^1 r_R} \right)^Q \frac{1}{V_{\frac{r_L}{r_R}}^L(x, \delta)} & \text{if } \frac{r_L}{r_R} \geq \frac{r_L^1}{r_R^1} \\ \left(\frac{r_L^1 r_R}{r_L r_R^1} \right)^Q \frac{1}{V_{\frac{r_L^1}{r_R^1}}^L\left(x, \frac{r_R r_L^1}{r_L} \delta\right)} & \text{if } \frac{r_L}{r_R} \leq \frac{r_L^1}{r_R^1} \end{cases}$$

In the case when $r_L \leq r_R$ but $r_L^1 \geq r_R^1$, we have

$$\frac{1}{V_x^R\left(\frac{r_R^1}{r_L^1}, \delta\right)} \lesssim \begin{cases} \left(\frac{r_L r_R^1}{r_L^1 r_R} \right)^Q \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{r_R^1}{r_L^1} \delta\right)} & \text{if } \frac{r_L}{r_R} \geq \frac{r_L^1}{r_R^1} \\ \left(\frac{r_L^1 r_R}{r_L r_R^1} \right)^Q \frac{1}{V_{\frac{r_L^1}{r_R^1}}^L\left(x, \frac{r_R}{r_L} \delta\right)} & \text{if } \frac{r_L}{r_R} \leq \frac{r_L^1}{r_R^1} \end{cases} \quad (23)$$

Proof. This follows by combining Corollary 4.10 with Lemma 5.7. \square

6 Bump Functions and Elementary Operators

We are now in a position to better understand the normalized bump functions from Definition 2.1 and the elementary kernels from Definition 2.4. The intuition for the normalized bump functions is easy to understand. Indeed, if ϕ and ψ are two k -normalized bump functions in the sense of Definition 3.1, then

$$\text{Ker} \left(\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\psi^{(r_R)} \right) \right) (x, \cdot)$$

is essentially an r_L, r_R normalized bump function centered at x of some order, dependent on k . This follows from (21) and Theorem 5.2. Following this analogy, we have:

Lemma 6.1. *Suppose ϕ_{r_L, r_R}^x is an r_L, r_R normalized bump function centered at x , and ψ is a C^∞ function supported in the unit ball B . Then,*

$$\text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x$$

is a constant times a $r_L \wedge r'_L, r_R$ normalized bump function, except perhaps with support on a ball with a constant times the radius of the support of a normalized bump function. The order of $\text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x$ will depend on the order of ϕ_{r_L, r_R}^x in a way implicit in the proof.

Proof. The support and bounds of $\text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x$ are easy to see. Indeed, fixing χ as in Remark 5.4, we see that

$$|\phi_{r_L, r_R}^x(z)| \lesssim \text{Ker} \left(\text{Op}_L \left(\chi^{(r_L)} \right) \text{Op}_R \left(\chi^{(r_R)} \right) \right) (z, x)$$

Thus,

$$\begin{aligned} \left| \text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z) \right| &\lesssim \text{Ker} \left(\text{Op}_L \left(\phi^{(r'_L)} \right) \text{Op}_L \left(\chi^{(r_L)} \right) \text{Op}_R \left(\chi^{(r_R)} \right) \right) (x, z) \\ &\lesssim \text{Ker} \left(\text{Op}_L \left(\tilde{\chi}^{(r_L \wedge r'_L)} \right) \text{Op}_R \left(\chi^{(r_R)} \right) \right) (x, z) \end{aligned}$$

where $\tilde{\chi}$ has some fixed bound and is supported in some fixed ball. Thus the bounds for $\text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z)$ follow from (21) and Theorem 5.2. It only remains to bound the derivatives of $\text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z)$.

For ∇_R derivatives, this is easy. Indeed,

$$\nabla_R^\alpha \text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z) = \text{Op}_L \left(\phi^{(r'_L)} \right) \nabla_R^\alpha \phi_{r_L, r_R}^x(z)$$

and then the result follows from the definition of a normalized bump function and our previous bounds. Similarly, if $r'_L \leq r_L$, we have:

$$\nabla_L^\alpha \text{Op}_L \left(\phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z) = r_L'^{|\alpha|} \text{Op}_L \left(\nabla_L^\alpha \phi^{(r'_L)} \right) \phi_{r_L, r_R}^x(z)$$

The only problem that remains is when $r_L \leq r'_L$. In that case we use the following result:

$$\nabla_L^\alpha \text{Op}_L \left(\phi^{(r'_L)} \right) = \sum_{|\beta|=|\alpha|} \text{Op}_L \left(\phi_\beta^{(r'_L)} \right) \nabla_L^\beta \quad (24)$$

where ϕ_β is of the same form as ϕ . From (24), ∇_L derivatives follow much in the same way as ∇_R derivatives. We leave the details to the reader. When one takes ∇_L and ∇_R derivatives simultaneously, the result follows from a combination of the above two methods.

(24) is well known, and so to save space and not introduce too much extra notation, we prove it only in the case of the Heisenberg group (see Section 9.1 for the notation used here). Indeed, suppose $\nabla_L = (X_L, Y_L)$ and $\nabla_R = (X_R, Y_R)$,

with $[X_L, Y_L] = 4\partial_t = -[X_R, Y_R]$, and $X_L = X_R + 4y\partial_t$. Then,

$$\begin{aligned}
X_L \text{Op}_L \left(\phi^{(r'_L)} \right) &= (X_R + 4yT) \text{Op}_L \left(\phi^{(r'_L)} \right) \\
&= \text{Op}_L \left(\phi^{(r'_L)} \right) X_L + 4 \frac{1}{r'_L} \text{Op}_L \left(y \phi^{(r'_L)} \right) \partial_t \\
&= \text{Op}_L \left(\phi^{(r'_L)} \right) X_L + \frac{1}{r'_L} \text{Op}_L \left(y \phi^{(r'_L)} \right) [X_L, Y_L] \\
&= \text{Op}_L \left(\phi^{(r'_L)} \right) X_L + \text{Op}_L \left(X_L y \phi^{(r'_L)} \right) Y_L - \text{Op}_L \left(Y_L y \phi^{(r'_L)} \right) X_L
\end{aligned}$$

The proof of the more general result follows in a similar fashion. \square

Theorem 6.2. *Suppose $\phi \in \mathcal{S}_0 \widehat{\otimes} \mathcal{S}_0$, then $\text{Op}_T \left(\phi^{(r_L, r_R)} \right)$ is an r_L, r_R elementary operator. Indeed, this is true uniformly as ϕ varies over a bounded set, and $r_L, r_R > 0$ vary.*

Proof. By Remark 3.11 it suffices to prove the result for elementary tensor products. Ie, we replace $\phi(x, y)$ with $\phi(x) \psi(y)$, where $\phi, \psi \in \mathcal{S}_0$. Thus we are concerned with showing:

$$\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\psi^{(r_R)} \right)$$

is an r_L, r_R elementary operator.

Since $\phi, \psi \in \mathcal{S}_0$, we may apply Lemma 3.7, for every $N_1, N_2, N_3, N_4 \in \mathbb{N}$, we may write,

$$\begin{aligned}
\phi &= \sum_{|\alpha_1|=N_1} \sum_{|\alpha_2|=N_2} \nabla_L^{\alpha_1} \nabla_R^{\alpha_2} \phi_{\alpha_1, \alpha_2} \\
\psi &= \sum_{|\beta_1|=N_3} \sum_{|\beta_2|=N_4} \nabla_R^{\beta_1} \nabla_L^{\beta_2} \psi_{\beta_1, \beta_2}
\end{aligned}$$

with $\phi_{\alpha_1, \alpha_2}, \psi_{\beta_1, \beta_2} \in \mathcal{S}_0$. Therefore,

$$\begin{aligned}
\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\psi^{(r_R)} \right) &= r_L^{-N_1-N_2} r_R^{-N_3-N_4} \\
&\times \sum_{\substack{|\alpha_1|=N_1, |\alpha_2|=N_2 \\ |\beta_1|=N_3, |\beta_2|=N_4}} (-1)^{|\alpha_2|+|\beta_2|} \nabla_L^{\alpha_2} \nabla_R^{\beta_1} \text{Op}_L \left(\phi_{\alpha_1, \alpha_2}^{(r_L)} \right) \text{Op}_L \left(\psi_{\beta_2, \beta_2}^{(r_R)} \right) \nabla_L^{\alpha_2} \nabla_R^{\beta_2}
\end{aligned}$$

and so to show that $\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\psi^{(r_R)} \right)$ is an elementary operator, it suffices to show that:

$$\text{Ker} \left(\text{Op}_L \left(\phi^{(r_L)} \right) \text{Op}_R \left(\psi^{(r_R)} \right) \right)$$

satisfies the estimates of (3). For this purpose, it will suffice to just assume $\phi, \psi \in \mathcal{S}$. Moreover, it is easy to reduce the problem to the case when $|\alpha_1| = |\alpha_2| = |\beta_1| = |\beta_2| = 0$, and so we prove it only in this case, leaving the details to the reader. Henceforth, we will only need that ϕ and ψ are rapidly decreasing.

It is easy to see that it suffices to prove (3) for $r_L = 2^{j_0}$, $r_R = 2^{k_0}$ for $j_0, k_0 \in \mathbb{Z}$ (a completely unnecessary reduction, but it makes notation a little easier). We also assume $k_0 \geq j_0$, the other situation being similar. Let $\chi_0(x) = \chi_B(2x)$ (B is $\|x\| < 1$), and $\chi_1(x) = \chi_0(x) - \chi_0(2x)$, so that:

$$\chi_0(x) + \sum_{j=1}^{\infty} \chi_1(2^{-j}x) = 1$$

Define ϕ_j by the equation:

$$2^{(-j)Q} \phi_j(2^{-j}x) = \begin{cases} \chi_1(2^{-j}x) \phi(x) & \text{if } j > 0, \\ \chi_0(x) \phi(x) & \text{if } j = 0 \end{cases}$$

and define ψ_k in a similar manner; so that:

$$\sum_{j=0}^{\infty} \phi_j(2^{-j}) = \phi$$

$$\sum_{k=0}^{\infty} \psi_k(2^{-k}) = \psi$$

and $\phi_j(x), \psi_k(x)$ are supported where $\|x\| \leq 1$. Using that ϕ is rapidly decreasing, we see that for any $N \in \mathbb{N}$:

$$\begin{aligned} \left| \phi_j^{(2^{j_0-j})}(x) \right| &\lesssim 2^{j_0 Q} (1 + 2^j)^{-N} \chi_{\{\|x\| \leq 2^{j-j_0}\}}(x) \\ &\lesssim (1 + 2^j)^{-N+Q} 2^{(j_0-j)Q} \chi_B(2^{j_0-j}x) \\ &= (1 + 2^j)^{-N'} \chi_B^{(2^{j_0-j})}(x) \end{aligned}$$

and similarly,

$$\left| \psi_k^{(2^{k_0-k})}(x) \right| \lesssim (1 + 2^k)^{-N'} \chi_B^{(2^{k_0-k})}(x)$$

As in Section 5.1, we will use the notation:

$$K_{r_L, r_R} = \text{Ker} \left(\text{Op}_L \left(\chi_B^{(r_L)} \right) \text{Op}_R \left(\chi_B^{(r_R)} \right) \right)$$

We are ready to compute our main bound (we use (21), Lemma 5.7, and Theo-

rem 5.2 freely below):

$$\begin{aligned}
& \text{Ker} \left(\text{Op}_L \left(\phi^{(2^{j_0})} \right) \text{Op}_R \left(\psi^{(2^{k_0})} \right) \right) (x, z) \\
&= \sum_{\substack{j \geq 0 \\ k \geq 0}} \text{Ker} \left(\text{Op}_L \left(\phi_j^{(2^{j_0-k})} \right) \text{Op}_R \left(\psi_k^{(2^{k_0-k})} \right) \right) (x, z) \\
&\lesssim \sum_{\substack{j \geq 0 \\ k \geq 0}} (1+2^j)^{-N_1} (1+2^k)^{-N_2} \text{Ker} \left(\text{Op}_L \left(\chi_B^{(2^{j_0-j})} \right) \text{Op}_R \left(\chi_B^{(2^{k_0-k})} \right) \right) (x, z) \\
&\lesssim \sum_{\substack{j \geq 0 \\ k \geq 0}} (1+2^j)^{-N_1} (1+2^k)^{-N_2} \begin{cases} 2^{(j-k)Q} K_{2^{j_0-j}, 2^{k_0-j}}(x, z) & \text{if } j \geq k \\ 2^{(k-j)Q} K_{2^{j_0-k}, 2^{k_0-k}}(x, z) & \text{if } k \geq j \end{cases} \\
&\lesssim \sum_{\substack{j \geq 0 \\ k \geq 0}} (1+2^j)^{-N_1} (1+2^k)^{-N_2} \begin{cases} 2^{(j-k)Q} \frac{\chi_{\left\{ \frac{\rho_{2^{j_0-k_0}}^L(x, z)}{V_{2^{j_0-k_0}}^L(x, 2^{j-j_0})} \leq 2^{j-j_0} \right\}}}{V_{2^{j_0-k_0}}^L(x, 2^{j-j_0})} & \text{if } j \geq k \\ 2^{(k-j)Q} \frac{\chi_{\left\{ \frac{\rho_{2^{j_0-k_0}}^L(x, z)}{V_{2^{j_0-k_0}}^L(x, 2^{k-j_0})} \leq 2^{k-j_0} \right\}}}{V_{2^{j_0-k_0}}^L(x, 2^{k-j_0})} & \text{if } k \geq j \end{cases}
\end{aligned}$$

Both the sum when $j \geq k$ and the sum when $k \geq j$ fall off faster than a geometric series (for N_1 and N_2 chosen sufficiently large), and therefore are bounded by their first term. For the sum when $j \geq k$, the first term is when k is zero and when $2^{j-j_0} \approx \rho_{2^{j_0-k_0}}^L(x, z)$ (or when $j = 0$ if such a j is less than 0), with a similar result when $k \geq j$. Hence, we see that:

$$\begin{aligned}
& \text{Ker} \left(\text{Op}_L \left(\phi^{(2^{j_0})} \right) \text{Op}_R \left(\psi^{(2^{k_0})} \right) \right) (x, z) \\
&\lesssim \frac{1}{(1+2^{j_0} \rho_{2^{j_0-k_0}}^L(x, z))^N V_{2^{j_0-k_0}}^L(x, 2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, z))}
\end{aligned}$$

for any $N \geq 0$, completing the proof. \square

Theorem 6.2 along with Proposition 3.15 show that every two-sided convolution operator can be decomposed as a sum of elementary kernels. In fact, this will be true for every operator in \mathcal{A}' ; moreover, this will characterize \mathcal{A}' . We devote the rest of this section to proving these facts. Our first step is an analog of Lemma 3.8.

Lemma 6.3. *Suppose $E_{2^{j_1}, 2^{k_1}}$ is a $2^{j_1}, 2^{k_1}$ elementary operator, and $E_{2^{j_2}, 2^{k_2}}$ is a $2^{j_2}, 2^{k_2}$ elementary operator. Then,*

$$E_{2^{j_1}, 2^{k_1}} E_{2^{j_2}, 2^{k_2}} = 2^{-|j_1-j_2|-|k_1-k_2|} E_{2^{j_3}, 2^{k_3}}$$

where j_3 can be either j_1 or j_2 and k_3 can be either k_1 or k_2 and $E_{2^{j_3}, 2^{k_3}}$ is a $2^{j_3}, 2^{k_3}$ elementary operator uniformly as j_1, j_2, k_1, k_2 vary over \mathbb{Z} with constants only depending on the constants for $E_{2^{j_1}, 2^{k_1}}$ and $E_{2^{j_2}, 2^{k_2}}$.

Proof. Let $\phi_{j_i, k_i} = \text{Ker}(E_{2^{j_i}, 2^{k_i}})$, $i = 1, 2$. Thus, we are interested in the function

$$\phi_{j_3, k_3}(x, z) = \int \phi_{j_1, k_1}(x, y) \phi_{j_2, k_2}(y, z) dy$$

Suppose, for a moment, that $j_2 \geq j_1$. Then we see, from Definition 2.4, that

$$\begin{aligned} \int \phi_{j_3, k_3}(x, z) &= \sum_{|\alpha|=N} 2^{-Nj_2} \int \phi_{j_1, j_2}(x, y) \nabla_{L, y}^\alpha \psi_{j_2, k_2, \alpha}(y, z) dy \\ &= \sum_{|\alpha|=N} 2^{N(j_1 - j_2)} \int \psi_{j_1, k_2, \alpha}(x, y) \phi_{j_2, k_2, \alpha}(y, z) dy \end{aligned}$$

where the $\psi_{j_i, k_i, \alpha}$ are uniformly $2^{j_i}, 2^{k_i}$ elementary kernels. Hence, it suffices to consider only terms of the form:

$$2^{N(j_1 - j_2)} \int \psi_{j_1, k_1}(x, y) \psi_{j_2, k_2}(y, z) dy$$

where N is any fixed large integer (which may depend on the semi-norm we wish to estimate). Doing the same argument for k_1, k_2 , we see that it suffices to consider only terms of the form:

$$2^{-N|j_1 - j_2| - N|k_1 - k_2|} \int \psi_{j_1, k_1}(x, y) \psi_{j_2, k_2}(y, z) dy$$

We proceed in the case when $k_1 \geq j_1$ and $k_2 \geq j_2$. The three other cases follow with only minor changes to the proof, and we leave those details to the interested reader. Define

$$\chi_l(r) = \begin{cases} 1 & \text{if } l > 0 \text{ and } 2^{l-1} \leq r < 2^l, \\ 0 & \text{if } l > 0 \text{ and } r \geq 2^l \text{ or } r < 2^{l-1}, \\ 1 & \text{if } l = 0 \text{ and } r < 1, \\ 0 & \text{if } l = 0 \text{ and } r \geq 1. \end{cases}$$

In the following, N_1, N_2 can be any two fixed large integers we choose, and we

let χ be as in Remark 5.4:

$$\begin{aligned}
& \left| \int \psi_{j_1, k_1}(x, y) \psi_{j_2, k_2}(y, z) dy \right| \\
& \leq \sum_{\substack{l_1 \geq 0 \\ l_2 \geq 0}} \left| \int \chi_{l_1}(2^{j_1} \rho_{2^{j_1-k_1}}^L(x, y)) \psi_{j_1, k_1}(x, y) \chi_{l_2}(\rho_{2^{j_2-k_2}}^L(y, z)) \psi_{j_2, k_2}(y, z) dy \right| \\
& \lesssim \sum \int \frac{\chi_{\{\rho_{2^{j_1-k_1}}^L(x, y) \leq 2^{l_1-j_1}\}} \chi_{\{\rho_{2^{j_2-k_2}}^L(y, z) \leq 2^{l_2-j_2}\}}}{2^{N_1 l_1} V_{2^{j_1-k_1}}^L(x, 2^{l_1-j_1}) 2^{N_2 l_2} V_{2^{j_2-k_2}}^L(x, 2^{l_2-j_2})} dy \\
& \lesssim \sum 2^{-N_1 l_1 - N_2 l_2} \int \text{Ker} \left(\text{Op}_L \left(\chi^{(2^{j_1-l_1})} \right) \text{Op}_R \left(\chi^{(2^{k_1-l_1})} \right) \right) (x, y) \\
& \quad \times \text{Ker} \left(\text{Op}_L \left(\chi^{(2^{j_2-l_2})} \right) \text{Op}_R \left(\chi^{(2^{k_2-l_2})} \right) \right) (y, z) dy \\
& = \sum 2^{-N_1 l_1 - N_2 l_2} \text{Ker} \left(\text{Op}_L \left(\chi^{(2^{j_1-l_1})} * \chi^{(2^{j_2-l_2})} \right) \text{Op}_R \left(\chi^{(2^{k_2-l_2})} * \chi^{(2^{k_1-l_1})} \right) \right) (x, z) \\
& = \sum 2^{-N_1 l_1 - N_2 l_2} \text{Ker} \left(\text{Op}_L \left(\tilde{\chi}^{(2^{(j_1-l_1) \wedge (j_2-l_2)})} \right) \text{Op}_R \left(\tilde{\chi}^{(2^{(k_1-l_1) \wedge (k_2-l_2)})} \right) \right) (x, z)
\end{aligned} \tag{25}$$

where $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ are non-negative bounded functions with support in a fixed bounded set, with these bounds independent of $j_1, j_2, k_1, k_2, l_1, l_2$. Note that we could have achieved the same left hand side for (25) in the three other cases where we allow $k_1 < j_1$ or $k_2 < j_2$ or both. We will now drop our assumption $k_1 \geq j_1, k_2 \geq j_2$, though we will return to it at the end.

Let \tilde{B} be a large fixed ball containing the support of $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$, and define:

$$\tilde{K}_{r_1, r_2}(x, z) = \text{Ker} \left(\text{Op}_L \left(\chi_{\tilde{B}}^{(r_1)} \right) \text{Op}_R \left(\chi_{\tilde{B}}^{(r_2)} \right) \right) (x, z)$$

so that we have:

$$\begin{aligned}
& 2^{-N|j_1-j_2|-N|k_1-k_2|} \left| \int \psi_{j_1, k_1}(x, y) \psi_{j_2, k_2}(y, z) dy \right| \\
& \lesssim 2^{-N|j_1-j_2|-N|k_1-k_2|} \sum_{\substack{l_1 \geq 0 \\ l_2 \geq 0}} 2^{-N_1 l_1 - N_2 l_2} \tilde{K}_{2^{(j_1-l_1) \wedge (j_2-l_2)}, 2^{(k_1-l_1) \wedge (k_2-l_2)}}(x, z)
\end{aligned} \tag{26}$$

We now proceed in proving the lemma in the case when $j_3 = j_1$ and $k_3 = k_1$. The case when $j_3 = j_2$ and $k_3 = k_2$ is completely symmetric. The remaining two cases follow by similar arguments, and we leave those proofs to the reader. We also work in the case when $k_1 \geq j_1$, the other case being symmetric.

We separate the RHS of (26) into 4 sums: depending on whether $j_1 - l_1 \leq j_2 - l_2$ and whether $k_1 - l_1 \leq k_2 - l_2$. The first case we deal with is the sum over those l_1 and l_2 such that $j_1 - l_1 \leq j_2 - l_2$ and $k_1 - l_1 \leq k_2 - l_2$. In this case, we need only take $N = 1$. In what follows, we will use (21), Lemma 5.7, and

Theorem 5.2 freely (indeed, we will use their analogs for \tilde{K} which follow from the methods in Section 5.1; note the \tilde{K} we use here differs slightly from the one in Section 5.1). We have:

$$\begin{aligned}
& 2^{-|j_1-j_2|-|k_1-k_2|} \sum_{l_1, l_2} 2^{-N_1 l_1 - N_2 l_2} \tilde{K}_{2^{j_1-l_1}, 2^{k_1-l_1}}(x, z) \\
& \lesssim 2^{-|j_1-j_2|-|k_1-k_2|} \sum_{l_1 \geq 0} 2^{-N_1 l_1} \tilde{K}_{2^{j_1-l_1}, 2^{k_1-l_1}}(x, z) \\
& \lesssim 2^{-|j_1-j_2|-|k_1-k_2|} \sum_{l_1 \geq 0} 2^{-N_1 l_1} \frac{\chi_{\{\rho_{2^{j_1-k_1}}^L(x, z) \lesssim 2^{l_1-j_1}\}}}{V_{2^{j_1-k_1}}^L(x, 2^{l_1-j_1})}
\end{aligned}$$

This sum falls off geometrically, and is therefore bounded by a multiple of its first term, which occurs when $\rho_{2^{j_1-k_1}}^L(x, z) \approx 2^{l_1-j_1}$, or when $l_1 = 0$ (whichever l_1 is greater). Thus, we have that this sum is:

$$\lesssim 2^{-|j_1-j_2|-|k_1-k_2|} \frac{1}{(1 + 2^{j_1} \rho_{2^{j_1-k_1}}^L(x, z))^{N_1} V_{2^{k_1-j_1}}^L(x, 2^{-j_1} + \rho_{2^{j_1-k_1}}^L(x, z))}$$

which is $2^{-|j_1-j_2|-|k_1-k_2|}$ times the bound for a $2^{j_1}, 2^{k_1}$ elementary kernel.

We now turn to the case when $j_2 - l_2 \leq j_1 - l_1$ and $k_2 - l_2 \leq k_1 - l_1$. As we estimate this case, we will use the fact that we may choose N to be large and this will allow us to absorb some terms by changing N . When we do this, we will replace N by N' and then by N'' , etc.

$$\begin{aligned}
& \sum_{l_1, l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_1 l_1 - N_2 l_2} \tilde{K}_{2^{j_2-l_2}, 2^{k_2-l_2}}(x, z) \\
& \lesssim \sum_{l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_2 l_2} \tilde{K}_{2^{j_2-l_2}, 2^{k_2-l_2}}(x, z) \\
& \lesssim \sum_{l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_2 l_2} \\
& \quad \times \begin{cases} 2^{(j_1-k_1+k_2-j_2)Q} \tilde{K}_{2^{j_2-l_2}, 2^{j_2-l_2+k_1-j_1}}(x, z) & \text{if } j_1 - k_1 \geq j_2 - k_2 \\ 2^{(j_2-k_2+k_1-j_1)Q} \tilde{K}_{2^{k_2-l_2+j_1-k_1}, 2^{k_2-l_2}}(x, z) & \text{if } j_1 - k_1 \leq j_2 - k_2 \end{cases} \\
& \lesssim \sum_{l_2} 2^{-N'|j_1-j_2|-N'|k_1-k_2|-N_2 l_2} \begin{cases} \tilde{K}_{2^{j_2-l_2}, 2^{j_2-l_2+k_1-j_1}}(x, z) \\ \tilde{K}_{2^{k_2-l_2+j_1-k_1}, 2^{k_2-l_2}}(x, z) \end{cases} \\
& \lesssim \sum_{l_2} 2^{-N'|j_1-j_2|-N'|k_1-k_2|-N_2 l_2} \begin{cases} \frac{\chi_{\{\rho_{2^{j_1-k_1}}^L(x, z) \lesssim 2^{l_2-j_2}\}}}{V_{2^{j_1-k_1}}^L(x, 2^{l_2-j_2})} \\ \frac{\chi_{\{\rho_{2^{j_1-k_1}}^L(x, z) \lesssim 2^{k_1-j_1+l_2-k_2}\}}}{V_{2^{j_1-k_1}}^L(x, 2^{k_1-j_1+l_2-k_2})} \end{cases}
\end{aligned}$$

This sum falls off geometrically, and is therefore bounded by its first term. Thus,

we have:

$$\begin{aligned}
& \sum_{l_1, l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_1 l_1-N_2 l_2} \tilde{K}_{2^{j_2-l_2}, 2^{k_2-l_2}}(x, z) \\
& \lesssim 2^{-N'|j_1-j_2|-N'|k_1-k_2|} \\
& \quad \times \begin{cases} \frac{1}{\left(1+2^{j_2} \rho_{2^{j_1-k_1}}^L(x, z)\right)^{N_2} V_{2^{j_1-k_1}}^L\left(x, 2^{-j_2}+\rho_{2^{j_1-k_1}}^L(x, z)\right)} \\ \frac{1}{\left(1+2^{k_2+j_1-k_1} \rho_{2^{j_1-k_1}}^L(x, z)\right)^{N_2} V_{2^{j_1-k_1}}^L\left(x, 2^{k_1-j_1-k_2}+\rho_{2^{j_1-k_1}}^L(x, z)\right)} \end{cases} \\
& \lesssim \frac{2^{-N''|j_1-j_2|-N''|k_1-k_2|}}{\left(1+2^{j_1} \rho_{2^{j_1-k_1}}^L(x, z)\right)^{N_2} V_{2^{j_1-k_1}}^L\left(x, 2^{-j_1}+\rho_{2^{j_1-k_1}}^L(x, z)\right)}
\end{aligned}$$

thereby completing the bound in this case.

We now turn to the case when $j_1 - l_1 \leq j_2 - l_2$ and $k_1 - l_1 \geq k_2 - l_2$.

$$\begin{aligned}
& \sum_{l_1, l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_1 l_1-N_2 l_2} \tilde{K}_{2^{j_1-l_1}, 2^{k_2-l_2}}(x, z) \\
& \lesssim \sum_{l_1, l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_1 l_1-N_2 l_2} 2^{(k_1-k_2+l_2-l_1)Q} \tilde{K}_{2^{j_1-k_1+k_2-l_2}, 2^{k_2-l_2}}(x, z)
\end{aligned}$$

where, when we applied Lemma 5.7, only the latter case ($k_1 - l_1 \geq k_2 - l_2$) applies. Continuing our bound, we have:

$$\lesssim \sum_{l_2} 2^{-N'|j_1-j_2|-N'|k_1-k_2|-N_2' l_2} \tilde{K}_{2^{j_1-k_1+k_2-l_2}, 2^{k_2-l_2}}(x, z)$$

but this is just the lower case for our computation when $k_1 - l_1 \geq k_2 - l_2$ and $j_1 - l_1 \geq j_2 - l_2$. Thus, we have:

$$\begin{aligned}
& \sum_{l_1, l_2} 2^{-N|j_1-j_2|-N|k_1-k_2|-N_1 l_1-N_2 l_2} \tilde{K}_{2^{j_1-l_1}, 2^{k_2-l_2}}(x, z) \\
& \lesssim \frac{2^{-N''|j_1-j_2|-N''|k_1-k_2|}}{\left(1+2^{j_1} \rho_{2^{j_1-k_1}}^L(x, z)\right)^{N_2'} V_{2^{j_1-k_1}}^L\left(x, 2^{-j_1}+\rho_{2^{j_1-k_1}}^L(x, z)\right)}
\end{aligned}$$

Finally, when $j_1 - l_1 \geq j_2 - l_2$ and $k_1 - l_1 \leq k_2 - l_2$, the proof proceeds as in the previous case, but now one ends up with the upper case for our computation when $k_1 - l_1 \geq k_2 - l_2$ and $j_1 - l_1 \geq j_2 - l_2$. Putting all of this together, we have:

$$\begin{aligned}
& \left| \int \phi_{j_1, k_1}(x, y) \phi_{j_2, k_2}(y, z) dy \right| \\
& \lesssim \frac{2^{-|j_1-j_2|-|k_1-k_2|}}{\left(1+2^{j_1} \rho_{2^{j_1-k_1}}^L(x, z)\right)^N V_{2^{j_1-k_1}}^L\left(x, 2^{-j_1}+\rho_{2^{j_1-k_1}}^L(x, z)\right)}
\end{aligned}$$

for any N we choose.

Now let's turn to derivatives. Fix ordered multi-indices $\alpha_1, \beta_1, \alpha_2, \beta_2$, and consider:

$$\begin{aligned} & \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} \int \phi_{j_1,k_1}(x,y) \phi_{j_2,k_2}(y,z) dy \right| \\ & \leq \sum \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} 2^{-|\alpha_2||k_1-k_2|-|\beta_2||j_1-j_2|} \int \psi_{j_1,k_1}(x,y) \psi_{j_2,k_2}(y,z) dy \right| \end{aligned}$$

where this is some finite sum and the ψ_{j_i,k_i} are elementary kernels, as we saw at the start of the proof. Applying the definition of elementary kernels, we see:

$$\begin{aligned} & \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} 2^{-|\alpha_2||k_1-k_2|-|\beta_2||j_1-j_2|} \int \psi_{j_1,k_1}(x,y) \psi_{j_2,k_2}(y,z) dy \right| \\ & = 2^{j_1(|\alpha_1|+|\alpha_2|)+k_1(|\beta_1|+|\beta_2|)} \left| \int \tilde{\psi}_{j_1,k_1}(x,y) \tilde{\psi}_{j_2,k_2}(y,z) dy \right| \end{aligned}$$

where the $\tilde{\psi}$ are also elementary kernels. Hence, our bound for the composition of two elementary kernels proved above, applied to:

$$\left| \int \tilde{\psi}_{j_1,k_1}(x,y) \tilde{\psi}_{j_2,k_2}(y,z) dy \right|$$

gives the proper bound from the definition of $2^{j_1}, 2^{k_1}$ elementary kernels for:

$$\left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,x}^{\beta_1} \nabla_{R,z}^{\beta_2} \int \phi_{j_1,k_1}(x,y) \phi_{j_2,k_2}(y,z) dy \right|$$

Finally, we need to see that we may “pull out” derivatives as in Definition 2.4. Pulling out x derivatives, works easily:

$$\begin{aligned} & \int \phi_{j_1,k_1}(x,y) \phi_{j_2,k_2}(y,z) dy \\ & = \sum_{\substack{|\alpha|=N_1 \\ |\beta|=N_3}} 2^{-j_1 N_1 - k_1 N_3} \nabla_{L,x}^{\alpha} \nabla_{R,x}^{\beta} \int \psi_{j_1,k_1,\alpha,\beta}(x,y) \phi_{j_2,k_2}(y,z) dy \end{aligned}$$

and is therefore $2^{-j_1 N_1 - k_1 N_3}$ times a sum of terms of the same form.

Pulling out z derivatives takes one more step. Indeed, fix N_2 and N_4 and suppose we wish to pull out z left derivatives of order N_2 and right derivatives of order N_4 (as in Definition 2.4). Then consider,

$$\begin{aligned} & \int \phi_{j_1,k_1}(x,y) \phi_{j_2,k_2}(y,z) dy \\ & = \sum 2^{-N_2|j_1-j_2|-N_4|k_1-k_2|} \int \psi_{j_1,k_1}(x,y) \psi_{j_2,k_2}(y,z) dy \\ & = \sum_{\substack{|\alpha|=N_2 \\ |\beta|=N_4}} \nabla_{L,z}^{\alpha} \nabla_{R,z}^{\beta} 2^{-N_2 j_1 - N_4 k_1} \int \psi_{j_1,k_1}(x,y) \psi_{j_2,k_2,\alpha,\beta}(y,z) dy \end{aligned}$$

where the sum above is, as usual, a finite sum over such terms, as we saw in the beginning of the proof. This completes the proof. \square

Lemma 6.4. *Suppose ϕ is an r_L, r_R elementary kernel, $r_L \leq r_R$. Then if we define $\psi^{(r_L)}(x) = \phi(x, 0)$, we have that $\psi \in \mathcal{S}_0$, uniformly for ϕ which are uniformly r_L, r_R elementary kernels, with constants independent of r_L, r_R . When $r_R \leq r_L$, we have: $\psi^{(r_R)}(x) = \phi(x, 0)$ yields ψ uniformly in \mathcal{S}_0 .*

Proof. This is a simple consequence of the definitions. \square

Corollary 6.5. *Suppose $\phi \in \mathcal{S}_0$, $l \in \mathbb{Z}$, and $E_{2^j, 2^k}$ is a $2^j, 2^k$ elementary kernel. Then,*

$$E_{2^j, 2^k} \phi^{(2^l)} = 2^{-|l-j|-|l-k|} \psi^{(2^l)} \quad (27)$$

where $\psi \in \mathcal{S}_0$. As ϕ and $E_{2^j, 2^k}$ range over bounded sets, so does ψ . Moreover, this is true uniformly in j, k, l .

Proof. Define $\tilde{\psi}^{(2^l)} = E_{2^j, 2^k} \phi^{(2^l)}$. First, let us see that it will suffice to show that $\tilde{\psi}$ is rapidly decreasing (uniformly, in the relevant parameters). Indeed, suppose we have that it is rapidly decreasing. First, let us see how to obtain the factor $2^{-|l-j|-|l-k|}$ in (27). Consider, in the case $l \leq j$,

$$\begin{aligned} E_{2^j, 2^k} \phi^{(2^l)} &= \sum_{|\alpha|=1} 2^{-j} \tilde{E}_{2^j, 2^k, \alpha} \nabla_L^\alpha \phi^{(2^l)} \\ &= 2^{l-j} \sum_{|\alpha|=1} \tilde{E}_{2^j, 2^k, \alpha} \tilde{\phi}_\alpha^{(2^l)} \end{aligned}$$

and so is a finite sum of terms of the same form, but now with a factor of 2^{l-j} out front.

On the other hand, if $j \leq l$, we may apply Lemma 3.7 to see:

$$\begin{aligned} E_{2^j, 2^k} \phi^{(2^l)} &= \sum_{|\alpha|=1} 2^{-l} E_{2^j, 2^k} \nabla_L^\alpha \tilde{\phi}_\alpha^{(2^l)} \\ &= \sum_{|\alpha|=1} 2^{j-l} \tilde{E}_{2^j, 2^k, \alpha} \tilde{\phi}_\alpha^{(2^l)} \end{aligned}$$

and so it is a finite sum of terms of the same form, but now with a factor of 2^{j-l} out front. In a similar manner we may obtain a factor of $2^{-|k-l|}$ out front.

Thus we have seen that, given that $\tilde{\psi}$ is rapidly decreasing, we have that ψ in the statement of the corollary, is rapidly decreasing, uniformly in the relevant parameters. Let us turn to derivatives of ψ . It is easy to see from the Definition 2.4 that if $|\alpha| = 1$,

$$\nabla_L^\alpha E_{2^j, 2^k} = \sum_{|\beta|=1} \tilde{E}_{2^j, 2^k} \nabla_L^\beta \quad (28)$$

Hence,

$$\begin{aligned}\nabla_L^\alpha E_{2^j, 2^k} \phi^{(2^l)} &= \sum_{|\beta|=1} \tilde{E}_{2^j, 2^k} \nabla_L^\beta \phi^{(2^l)} \\ &= 2^l \sum_{|\beta|=1} \tilde{E}_{2^j, 2^k} \tilde{\phi}_\beta^{(2^l)}\end{aligned}$$

and so is 2^l times a finite sum of terms of the same form. Thus, ψ behaves properly under derivatives, and we have shown that ψ is uniformly in \mathcal{S} .

To see ψ is uniformly in \mathcal{S}_0 , we need to “pull out” derivatives. However, we merely use the other direction of (28) to see:

$$\begin{aligned}E_{2^j, 2^k} \phi^{(2^l)} &= 2^{-l} \sum_{|\alpha|=1} E_{2^j, 2^k} \nabla_L^\alpha \tilde{\phi}_\alpha^{(2^l)} \\ &= \sum_{|\alpha|=1} \sum_{|\beta|=1} 2^{-l} \nabla_L^\beta \tilde{E}_{2^j, 2^k, \beta} \tilde{\phi}_\alpha^{(2^l)}\end{aligned}$$

and so one can “pull out” derivatives.

Thus, we turn to proving that $\tilde{\psi}$ is rapidly decreasing. In fact, by the argument earlier in this proof, it suffices to show that for each M , there exists an N such that:

$$2^{-|l-j|N-|l-k|N} \left| \tilde{\psi}^{(2^l)}(x) \right| \lesssim 2^{lQ} (1 + 2^l |x|)^{-M}$$

And therefore, it suffices to show that if $l_0 = \min\{l, j, k\}$, then

$$\left| E_{2^j, 2^k} \phi(x) \right| \lesssim 2^{l_0 Q} (1 + 2^{l_0} |x|)^{-M}$$

to do this, we will show that if we redefine $\tilde{\psi}$ to be:

$$\tilde{\psi}^{(2^{l_0})} = E_{2^j, 2^k} \phi^{(2^l)}$$

then we have that $\tilde{\psi} \in \mathcal{S}_0$ uniformly in the relevant parameters. We proceed in the cases when $l_0 = j$ or $l_0 = l$. The case when $l_0 = k$ is similar to that when $l_0 = j$.

We will next prove that

$$E_{2^j, 2^k} \text{Op}_L \left(\phi^{(2^l)} \right) = 2^{-|j-l|} E_{2^{l_0}, 2^k}$$

where $E_{2^{l_0}, 2^k}$ is a $2^{l_0}, 2^k$ elementary kernel. Then, the result will follow from the fact that:

$$E_{2^j, 2^k} \phi^{(2^l)} = \text{Ker} \left(E_{2^j, 2^k} \text{Op}_L \left(\phi^{(2^l)} \right) \right) (\cdot, 0)$$

and applying Lemma 6.4.

We consider the identity operator I as a right convolution operator. Then, we may apply Proposition 3.3 to see that:

$$I = \sum_{k_2} \text{Op}_R \left(\psi_{k_2}^{(2^{k_2})} \right)$$

with this sum converging strongly in L^2 . It is easy to see that everything we're dealing with in this proof is continuous on L^2 , hence,

$$\begin{aligned} E_{2^j, 2^k} \text{Op}_L \left(\phi^{(2^l)} \right) &= E_{2^j, 2^k} I \text{Op}_L \left(\phi^{(2^l)} \right) \\ &= \sum_{k_2} E_{2^j, 2^k} \text{Op}_R \left(\psi_{k_2}^{(2^{k_2})} \right) \text{Op}_L \left(\phi^{(2^l)} \right) \\ &= \sum_{k_2} E_{2^j, 2^k} E_{2^l, 2^{k_2}} \\ &= \sum_{k_2} 2^{-|j-l|-|k-k_2|} E_{2^{l_0}, 2^k} \\ &= 2^{-|j-l|} E_{2^{l_0}, 2^k} \end{aligned}$$

where we have used Lemma 6.3 and Theorem 6.2, completing the proof. \square

Theorem 6.6. *Suppose for each $j, k \in \mathbb{Z}$ we have $E_{2^j, 2^k}$ a $2^j, 2^k$ elementary operator, uniformly in j, k . Then,*

$$T = \sum_{j, k} E_{2^j, 2^k} \tag{29}$$

converges in the topology of bounded convergence as operators $\mathcal{S}_0 \rightarrow \mathcal{S}_0$, and also converges in the strong operator topology as bounded operators $L^2 \rightarrow L^2$. Moreover, $T \in \mathcal{A}'$. Conversely, every operator in \mathcal{A}' can be decomposed as in (29).

Proof. The convergence of the sum

$$T = \sum_{j, k} E_{2^j, 2^k}$$

in the topology of bounded convergence as operators $\mathcal{S}_0 \rightarrow \mathcal{S}_0$ follows directly from Corollary 6.5, thinking of a fixed element $\phi \in \mathcal{S}_0$ as $\phi = \phi^{(2^0)}$. To see that the sum (29) converges in the strong operator topology $L^2 \rightarrow L^2$, we apply the Cotlar-Stein lemma. Indeed, the adjoint of a $2^j, 2^k$ elementary operator is again a $2^j, 2^k$ elementary operator (see Remark 2.8), and therefore,

$$E_{2^{j_1}, 2^{k_1}}^* E_{2^{j_2}, 2^{k_2}} = 2^{-|j_1-j_2|-|k_1-k_2|} \tilde{E}_{2^{j_2}, 2^{k_2}}$$

by Lemma 6.3. Thus, to see that the sum converges in the strong operator topology $L^2 \rightarrow L^2$, it suffices to show that the operators $\tilde{E}_{2^{j_2}, 2^{k_2}}$ are uniformly

bounded $L^2 \rightarrow L^2$. This follows easily from Lemma 8.1. Alternatively, it is easy to see that the $\tilde{E}_{2^{j_2}, 2^{k_2}}$ are uniformly NIS operators corresponding to the metrics $\rho_{2^{j_2}-k_2}^L$ (or $\rho_{2^{k_2}-j_2}^R$ if $k_2 \leq j_2$), which in turn correspond to spaces which are uniformly spaces of homogeneous type in the sense of [DJS85], by the remarks in Section 4. Thus the uniform L^2 boundedness for $\tilde{E}_{2^{j_2}, 2^{k_2}}$ follows by usual proofs that NIS operators are bounded on L^2 (see [Koe02, NRS01]).

To see $T \in \mathcal{A}'$, note that:

$$\begin{aligned} TE_{2^{j_0}, 2^{k_0}} &= \sum_{j,k} E_{2^j, 2^k} E_{2^{j_0}, 2^{k_0}} \\ &= \sum_{j,k} 2^{-|j-j_0|-|k-k_0|} E_{2^{j_0}, 2^{k_0}} \\ &= E_{2^{j_0}, 2^{k_0}} \end{aligned}$$

where we have applied Lemma 6.3, where $E_{2^{j_0}, 2^{k_0}}$ is a $2^{j_0}, 2^{k_0}$ elementary operator. For more general r_L, r_R elementary operators, merely think of an r_L, r_R elementary operator as a $2^{j_0}, 2^{k_0}$ elementary operator, where j_0, k_0 are chosen to minimize $|2^{j_0} - r_L| + |2^{k_0} - r_R|$.

For the converse, suppose $T \in \mathcal{A}$. Thinking of the identity I as a two-sided convolution operator, we may apply Proposition 3.15 to see that:

$$I = \sum_{j,k \in \mathbb{Z}} \text{Op}_T \left(\phi_{j,k}^{(2^j, 2^k)} \right)$$

where $\{\phi_{j,k}\} \subset \mathcal{S}_0 \hat{\otimes} \mathcal{S}_0$ is a bounded set. Applying Theorem 6.2, we see:

$$I = \sum_{j,k \in \mathbb{Z}} E_{2^j, 2^k}$$

(where this sum converges strongly in L^2 , and as we have seen earlier in the proof, in the topology of bounded convergence $\mathcal{S}_0 \rightarrow \mathcal{S}_0$). Hence, we see:

$$T = TI = \sum_{j,k \in \mathbb{Z}} TE_{2^j, 2^k} = \sum_{j,k \in \mathbb{Z}} \tilde{E}_{2^j, 2^k}$$

completing the proof. \square

Corollary 6.7. *Let K be a product kernel. Then, $\text{Op}_T(K) \in \mathcal{A}'$.*

Proof. This is a combination of Proposition 3.15 and Theorems 6.2 and 6.6. \square

7 Equivalence of \mathcal{A} and \mathcal{A}'

In this section, we show that \mathcal{A} and \mathcal{A}' are the same spaces. To begin with, we will need a better understanding of the function B defined in Section 2 by:

$$B(r_L, r_R, N_L, N_R, m, x, y) = r_L^{N_L} r_R^{N_R} \left(1 + r_L \rho_{\frac{r_L}{r_R}}^L(x, y)\right)^{-m} \frac{1}{V_{\frac{r_L}{r_R}}^L\left(x, \frac{1}{r_L} + \rho_{\frac{r_L}{r_R}}^L(x, y)\right)}$$

Lemma 7.1. *If $N_L, N_R \geq Q + m + 1$, we have,*

$$\sum_{\substack{j \leq j_0 \\ k \leq k_0}} B(2^j, 2^k, N_L, N_R, m, x, y) \approx B(2^{j_0}, 2^{k_0}, N_L, N_R, m, x, y) \quad (30)$$

and as a simple corollary:

$$\sum_{j \leq j_0} B(2^j, 2^{k_0}, N_L, N_R, m, x, y) \approx B(2^{j_0}, 2^{k_0}, N_L, N_R, m, x, y)$$

Proof. \gtrsim is clear, and so we focus on \lesssim . Without loss of generality, we may assume $j_0 \leq k_0$. We separate the sum (30) into the sum when $j \leq k$ and the sum when $k \leq j$. We consider, first, the easier case when $j \leq k$:

$$\begin{aligned} \sum_{\substack{j \leq k \\ j \leq j_0 \\ k \leq k_0}} B(j, k, N_L, N_R, m, x, y) &= \sum 2^{N_L j + N_R k} \frac{(1 + 2^j \rho_{2^{j-k}}^L(x, y))^{-m}}{V_{2^{j-k}}^L(x, 2^{-j} + \rho_{2^{j-k}}^L(x, y))} \\ &= \sum 2^{N_L j + N_R k} \frac{(1 + 2^{j-j_0} 2^{j_0} \rho_{2^{j_0-k_0} 2^{k_0-k+j-j_0}}^L(x, y))^{-m}}{V_{2^{j-k}}^L(x, 2^{j_0-j} 2^{-j_0} + \rho_{2^{j_0-k_0} 2^{k_0-k+j-j_0}}^L(x, y))} \end{aligned}$$

we now use the elementary fact that $\frac{1}{2} \rho_\epsilon^L \leq \rho_{2\epsilon}^L \leq \rho_\epsilon^L$, to see:

$$\lesssim \sum 2^{N_L j + N_R k} \frac{(1 + 2^{j-j_0+k-k_0} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j-k}}^L(x, 2^{k-k_0} (2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, y)))}$$

Applying Corollary 5.8, we have:

$$\lesssim \begin{cases} \sum 2^{N_L j + N_R k + (j_0-j+k-k_0)Q} \frac{(1 + 2^{j-j_0+k-k_0} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j_0-k_0}}^L(x, 2^{k-k_0} (2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, y)))} & \text{if } j_0 - k_0 \geq j - k \\ \sum 2^{N_L j + N_R k + (j-j_0+k_0-k)Q} \frac{(1 + 2^{j-j_0+k-k_0} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j_0-k_0}}^L(x, 2^{j-j_0} (2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, y)))} & \text{if } j_0 - k_0 \leq j - k \end{cases}$$

But for $\delta < 1$, $V_{2^{j-k}}^L(x, \delta r) \gtrsim \delta^Q V_{2^{j-k}}^L(x, r)$ (this is a consequence of Theorem 4.9), and so we have,

$$\begin{aligned} &\lesssim \sum 2^{N_L j + N_R k + (j_0-j+k_0-k)Q} \frac{(1 + 2^{j-j_0+k-k_0} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j_0-k_0}}^L(x, 2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, y))} \\ &\lesssim B(2^{j_0}, 2^{k_0}, N_L, N_R, m, x, y) \end{aligned}$$

since this sum is geometric, provided $N_L, N_R \geq Q + m + 1$.

We now turn to the sum when $k \leq j$:

$$\sum_{\substack{k \leq j \\ j \leq j_0 \\ k \leq k_0}} B(j, k, N_L, N_R, m, x, y) = \sum 2^{N_L j + N_R k} \frac{(1 + 2^k \rho_{2^{k-j}}^R(x, y))^{-m}}{V_{2^{k-j}}^R(x, 2^{-k} + \rho_{2^{k-j}}^R(x, y))} \quad (31)$$

Using the fact that $\rho_\epsilon^R = \epsilon \rho_\frac{1}{\epsilon}^L$ (here we have removed the restriction $\epsilon \leq 1$) we see:

$$\rho_{2^{k-j}}^R = \rho_{2^{k_0-j_0} 2^{j_0-j+k-k_0}}^R \geq 2^{j_0-j} \rho_{2^{k_0-j_0}}^R = 2^{k_0-j} \rho_{2^{j_0-k_0}}^L$$

Plugging this into (31), we see that (31) is

$$\lesssim \sum 2^{N_L j + N_R k} \frac{(1 + 2^{k_0-j_0+k-j} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{k-j}}^R(x, 2^{-k} + 2^{k_0-j} \rho_{2^{j_0-k_0}}^L(x, y))}$$

using that $j_0 \leq k_0$,

$$\lesssim \sum 2^{N_L j + N_R k} \frac{(1 + 2^{j_0-j} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{k-j}}^R(x, 2^{-k} + 2^{k_0-j} \rho_{2^{j_0-k_0}}^L(x, y))}$$

Applying Corollary 5.8 and using the fact that the indices we are summing over satisfy $k_0 - k \geq j_0 - j$ and so we are in the lower case of (23), and thus

$$\lesssim \sum 2^{N_L j + N_R k + (k_0 - k + j - j_0)Q} \frac{(1 + 2^{j_0-j} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j_0-k_0}}^L(x, 2^{k_0-k} 2^{-j_0} + 2^{k_0-j+k_0-j_0} \rho_{2^{j_0-k_0}}^L(x, y))}$$

using that $j \leq j_0 \leq k_0$, we see:

$$\begin{aligned} &\lesssim \sum 2^{N_L j + N_R k + (k_0 - k)Q} \frac{(1 + 2^{j_0-j} 2^{j_0} \rho_{2^{j_0-k_0}}^L(x, y))^{-m}}{V_{2^{j_0-k_0}}^L(x, 2^{-j_0} + \rho_{2^{j_0-k_0}}^L(x, y))} \\ &\lesssim B(j_0, k_0, N_L, N_R, m, x, y) \end{aligned}$$

provided $N_L, N_R \geq Q + m + 1$, completing the proof of the first estimate. The second estimate follows as a simple corollary. \square

Remark 7.2. In our proof that $\mathcal{A}' \subseteq \mathcal{A}$, we will see that the only reason we need to take N_0 large in Definition 2.2 is so that we may apply Lemma 7.1. Because of this, once we show that $\mathcal{A} = \mathcal{A}'$, we will see that we may replace N_0 in Definition 2.2 by $Q + m + 1$.

Lemma 7.3. *Suppose $\phi_{2^{j_1}, 2^{k_1}}^x$ is a normalized $2^{j_1}, 2^{k_1}$ bump function of order 0 centered at x , $\phi_{2^{j_2}, 2^{k_2}}^z$ is a normalized $2^{j_2}, 2^{k_2}$ bump function of order 0 centered at z , and $E_{2^{j_3}, 2^{k_3}}$ is a $2^{j_3}, 2^{k_3}$ elementary operator. Then,*

$$\begin{aligned} &\left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, E_{2^{j_3}, 2^{k_3}} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\ &\lesssim B(2^{j_1 \wedge j_2 \wedge j_3}, 2^{k_1 \wedge k_2 \wedge k_3}, 0, 0, m, x, z) \end{aligned}$$

with constants uniform in all the relevant parameters.

Proof. Let $j_0 = j_1 \wedge j_2$, and $k_0 = k_1 \wedge k_2$. We prove the result in the case when $k_3 - k_0 \geq j_3 - j_0$, the other case being similar. We also proceed in the case when $k_3 \geq j_3$, though the proof is essentially independent of this choice. Consider, letting χ be as in Remark 5.4,

$$\begin{aligned}
& \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, E_{2^{j_3}, 2^{k_3}} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\
& \lesssim \sum_{l \geq 0} \int_{\rho_{2^{j_3}-k_3}^L(y_1, y_2) \approx 2^{l-j_3}} \phi_{2^{j_1}, 2^{k_1}}^x(y_1) 2^{-lN} \frac{1}{V_{2^{j_3}-k_3}^L(y_1, 2^{l-j_3})} \phi_{2^{j_2}, 2^{k_3}}^z(y_2) dy_1 dy_2 \\
& \lesssim \sum_{l \geq 0} 2^{-lN} \text{Ker} \left(\text{Op}_L \left(\chi^{(2^{j_1})} \right) \text{Op}_R \left(\chi^{(2^{k_1})} \right) \text{Op}_L \left(\chi^{(2^{j_3-l})} \right) \text{Op}_R \left(\chi^{(2^{k_3-l})} \right) \right. \\
& \quad \left. \times \text{Op}_L \left(\chi^{(2^{j_2})} \right) \text{Op}_R \left(\chi^{(2^{k_2})} \right) \right) (x, z)
\end{aligned} \tag{32}$$

at this point we may drop the assumption $k_3 \geq j_3$, and note that we could have just as easily shown (32) in the case $k_3 \leq j_3$. In the above N is any fixed integer we choose, obtained from the rapid decrease of $\text{Ker}(E_{2^{j_3}, 2^{k_3}})$. Rearranging terms, and using that, for instance,

$$\chi^{(2^{j_1})} * \chi^{(2^{j_3-l})} * \chi^{(2^{j_2})} = \tilde{\chi}^{(2^{j_0 \wedge (j_3-l)})}$$

where $\tilde{\chi}$ is a bounded function of bounded support, with bounds independent of all the relevant parameters above, we see that the left hand side of (32) is

$$\begin{aligned}
& \lesssim \sum_{l \geq 0} 2^{-lN} \text{Ker} \left(\text{Op}_L \left(\tilde{\chi}^{(2^{j_0 \wedge (j_3-l)})} \right) \text{Op}_R \left(\tilde{\chi}^{(2^{k_0 \wedge (k_3-l)})} \right) \right) (x, z) \\
& \lesssim \sum_{l \geq 0} 2^{-lN} B \left(2^{j_0 \wedge (j_3-l)}, 2^{k_0 \wedge (k_3-l)}, 0, 0, m, x, z \right)
\end{aligned}$$

where we have applied (21) and Theorem 5.2. We separate this sum into three sums. The first:

$$\sum_{0 \leq l \leq k_3 - k_0} 2^{-lN} B(2^{j_0}, 2^{k_0}, 0, 0, m, x, z) \lesssim B(2^{j_0}, 2^{k_0}, 0, 0, m, x, z)$$

with this sum = 0 if $k_0 > k_3$. The second:

$$\begin{aligned}
& \sum_{[(k_3 - k_0) \vee 0] \leq l \leq j_3 - j_0} 2^{-lN} B(2^{j_0}, 2^{k_3-l}, 0, 0, m, x, z) \\
& = \sum_{[(k_3 - k_0) \vee 0] \leq l \leq j_3 - j_0} 2^{-k_3 N} B(2^{j_0}, 2^{k_3-l}, 0, N, m, x, z) \\
& \approx 2^{-k_3 N} B(2^{j_0}, 2^{k_3 \wedge k_0}, 0, N, m, x, z) \\
& \lesssim B(2^{j_0}, 2^{k_3 \wedge k_0}, 0, 0, m, x, z)
\end{aligned}$$

with this sum = 0 if $j_0 > j_3$, and we have applied Lemma 7.1 and we have used that we may take N large. Finally,

$$\begin{aligned}
& \sum_{[(j_3-j_0) \vee 0] \leq l} 2^{-lN} B(2^{j_3-l}, 2^{k_3-l}, m, 0, 0, x, y) \\
&= \sum_{[(j_3-j_0) \vee 0] \leq l} 2^{-j_3N/2-k_3N/2} B\left(2^{j_3-l}, 2^{k_3-l}, m, \frac{N}{2}, \frac{N}{2}, m, x, y\right) \\
&\leq \sum_{\substack{[(j_3-j_0) \vee 0] \leq l_1 \\ [(k_3-k_0) \vee 0] \leq l_2}} 2^{-j_3N/2-k_3N/2} B\left(2^{j_3-l_1}, 2^{k_3-l_2}, \frac{N}{2}, \frac{N}{2}, m, x, y\right) \\
&\lesssim 2^{-j_3N/2-k_3N/2} B\left(2^{j_3 \wedge j_0}, 2^{k_3 \wedge k_0}, \frac{N}{2}, \frac{N}{2}, m, x, y\right) \\
&\lesssim B(2^{j_3 \wedge j_0}, 2^{k_3 \wedge k_0}, 0, 0, m, x, y)
\end{aligned}$$

where again we have taken N large and applied Lemma 7.1. \square

Corollary 7.4. *Suppose $\phi_{2^{j_1}, 2^{k_1}}^x$ is a normalized $2^{j_1}, 2^{k_1}$ bump function centered at x , $\phi_{2^{j_2}, 2^{k_2}}^z$ is a normalized $2^{j_2}, 2^{k_2}$ bump function centered at z (each of some large order, how large will be implicit in the proof), and $E_{2^{j_3}, 2^{k_3}}$ is a $2^{j_3}, 2^{k_3}$ elementary operator. Then,*

$$\begin{aligned}
& \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\
& \lesssim 2^{((j_1 \wedge j_2) - j_3) \wedge 0 + ((k_1 \wedge k_2) - k_3) \wedge 0} \\
& \quad \times B(2^{j_1 \wedge j_2 \wedge j_3}, 2^{k_1 \wedge k_2 \wedge k_3}, |\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2|, m, x, z)
\end{aligned}$$

with constants uniform in all the relevant parameters.

Proof. Let $j_0 = j_1 \wedge j_2$ and $k_0 = k_1 \wedge k_2$. We first prove the result without the factor of:

$$2^{(j_0 - j_3) \wedge 0 + (k_0 - k_3) \wedge 0}$$

Suppose that $j_2 = j_0 \wedge j_3$. Then, we have:

$$\begin{aligned}
& \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \\
&= \sum_{|\alpha_3| = |\alpha_1|} \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_R^{\beta_1} \tilde{E}_{2^{j_3}, 2^{k_3}, \alpha_3} \nabla_L^{\alpha_3} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \\
&= 2^{j_2(|\alpha_1| + |\alpha_2|)} \sum_{|\alpha_3| = |\alpha_1|} \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_R^{\beta_1} \tilde{E}_{2^{j_3}, 2^{k_3}, \alpha_3} \nabla_R^{\beta_2} \tilde{\phi}_{2^{j_2}, 2^{k_2}, \alpha_3}^z \right\rangle_{L^2}
\end{aligned}$$

a finite sum of terms of the same form but with $|\alpha_1| = 0 = |\alpha_2|$, times $2^{j_2(|\alpha_1| + |\alpha_2|)}$. We get a similar result when $j_1 = j_0 \wedge j_3$. Finally, when $j_3 = j_3 \wedge j_0$, we merely let all the ∇_L derivatives land on the $E_{2^{j_3}, 2^{k_3}}$,

$$\begin{aligned}
& \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \\
&= 2^{j_3(|\alpha_1| + |\alpha_2|)} \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_R^{\beta_1} \tilde{E}_{2^{j_3}, 2^{k_3}} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2}
\end{aligned}$$

Doing a similar proof with the ks , we see:

$$\begin{aligned} & \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\ & \leq 2^{j_0 \wedge j_3 (|\alpha_1| + |\alpha_2|) k_0 \wedge k_3 (|\beta_1| + |\beta_2|)} \sum \left\langle \tilde{\phi}_{2^{j_1}, 2^{k_1}}^x, \tilde{E}_{2^{j_3}, 2^{k_3}} \tilde{\phi}_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \end{aligned}$$

where the sum denotes a finite sum of such terms. Applying Lemma 7.3, we see:

$$\begin{aligned} & \lesssim 2^{j_0 \wedge j_3 (|\alpha_1| + |\alpha_2|) k_0 \wedge k_3 (|\beta_1| + |\beta_2|)} B(2^{j_0 \wedge j_3}, 2^{k_0 \wedge k_3}, 0, 0, m, x, z) \\ & = B(2^{j_0 \wedge j_3}, 2^{k_0 \wedge k_3}, |\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2|, m, x, z) \end{aligned}$$

Which completes the proof, without the factor of $2^{(j_0 - j_3) \wedge 0 + (k_0 - k_3) \wedge 0}$. To see how to obtain that factor, suppose we are in the case when $j_0 = j_1 \leq j_3$. Then we “pull derivatives out” of $E_{2^{j_3}, 2^{k_3}}$ and let them land on $\phi_{2^{j_1}, 2^{k_1}}^x$; indeed,

$$\begin{aligned} & \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \\ & = \sum_{|\alpha|=1} 2^{-j_3} \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_L^{\alpha} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \\ & = \sum_{|\alpha_3|=|\alpha_1|} 2^{j_1 - j_3} \left\langle \tilde{\phi}_{2^{j_1}, 2^{k_1}, \alpha_3}^x, \nabla_L^{\alpha_3} \nabla_R^{\beta_1} E_{2^{j_3}, 2^{k_3}} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \end{aligned}$$

which is $2^{j_1 - j_3} = 2^{j_0 - j_3}$ times a finite sum of terms of the original form. A similar proof works for when $j_2 = j_0$ and for the ks . \square

Theorem 7.5. *Suppose $T \in \mathcal{A}'$, then $T \in \mathcal{A}$.*

Proof. We apply Theorem 6.6 to decompose T :

$$T = \sum_{j, k \in \mathbb{Z}} E_{2^j, 2^k}$$

where $E_{2^j, 2^k}$ are uniformly $2^j, 2^k$ elementary operators, and this sum converges in the strong operator topology as operators $L^2 \rightarrow L^2$. Fix m and fix $N_L, N_R \geq Q + m + 1$. Suppose $\phi_{2^{j_1}, 2^{k_1}}^x$ is a normalized $2^{j_1}, 2^{k_1}$ bump function centered at x , $\phi_{2^{j_2}, 2^{k_2}}^z$ is a normalized $2^{j_2}, 2^{k_2}$ bump function centered at z (each of some large order), and suppose that $|\alpha_1| + |\alpha_2| = N_L$, $|\beta_1| + |\beta_2| = N_R$. Then, letting $j_0 = j_1 \wedge j_2$, $k_0 = k_1 \wedge k_2$, we see:

$$\begin{aligned} & \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\ & \leq \sum_{j, k} \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} E_{2^j, 2^k} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^z \right\rangle_{L^2} \right| \\ & \lesssim \sum_{j, k} 2^{(j_0 - j) \wedge 0 + (k_0 - k) \wedge 0} B(2^{j_0 \wedge j}, 2^{k_0 \wedge k}, N_L, N_R, m, x, z) \\ & \lesssim B(2^{j_0}, 2^{k_0}, N_L, N_R, m, x, z) \end{aligned}$$

where we have applied Lemma 7.1 to get the last line, completing the proof. \square

We now turn to showing that $\mathcal{A} \subset \mathcal{A}'$. Fix $T \in \mathcal{A}$. Then, we wish to show that $TE_{r_L, r_R} = \tilde{E}_{r_L, r_R}$. As we have seen before, it will suffice to prove this result for $r_L = 2^j$, $r_R = 2^k$. This follows by choosing j, k to minimize $|r_L - 2^j| + |r_R - 2^k|$.

Lemma 7.6. *Given $r_L, r_R > 0$, $y \in G$, $m \geq 0$, there exists a N_0 such that if $N_L, N_R \geq N_0$, and $|\alpha_1| + |\alpha_2| = N_L$, $|\beta_1| + |\beta_2| = N_R$, and ϕ_{r_L, r_R}^y is a r_L, r_R bump function centered at y ,*

$$\left| \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{r_L, r_R}^y(x) \right| \lesssim B(r_L, r_R, N_L, N_R, m, x, y)$$

Proof. This follows directly from Definition 2.2, by taking $\phi_{r_L^{(1)}, r_R^{(1)}} \rightarrow \delta_x$, by taking $r_L^{(1)}, r_R^{(1)} \rightarrow \infty$.

To see that we can do this, merely take ϕ supported in the unit ball B such that $\int \phi = 1$. Then,

$$\text{Op}_L(\phi^{(2^j)}) \rightarrow I$$

as $j \rightarrow \infty$; similarly for $\text{Op}_R(\phi^{(2^k)})$. Thus if we set:

$$\phi_{2^j, 2^k}^x(z) = \text{Ker} \left(\text{Op}_L(\phi^{(2^j)}) \text{Op}_R(\phi^{(2^k)}) \right) (x, z)$$

we see that $\phi_{2^j, 2^k}^x \rightarrow \delta_x$. Since we saw in Section 6 $\phi_{2^j, 2^k}^x$ is essentially an normalized bump function (it may really have support in a ball with radius a constant factor times the ball it is supposed to be supported in, and need to be multiplied by a constant, but these only affect the answer by a constant), we are done. \square

Proposition 7.7. *$\text{Ker}(TE_{2^j, 2^k})$ satisfies the estimates (3) with $r_L = 2^j$, $r_R = 2^k$, uniformly in the relevant parameters.*

Proof. We proceed in the case when $j \leq k$, the other case being similar. Consider,

$$\begin{aligned} & \nabla_{L,x}^{\alpha_1} \nabla_{L,x}^{\beta_1} \nabla_{L,z}^{\alpha_2} \nabla_{L,z}^{\beta_2} \text{Ker}(TE_{2^j, 2^k})(x, z) \\ &= \nabla_{L,x}^{\alpha_1} \nabla_{L,x}^{\beta_1} (-1)^{|\alpha_2| + |\beta_2|} \text{Ker}(TE_{2^j, 2^k} \nabla_L^{\alpha_2} \nabla_R^{\beta_2})(x, z) \\ &= 2^{j|\alpha_2| + k|\beta_2|} \nabla_{L,x}^{\alpha_1} \nabla_{L,x}^{\beta_1} \text{Ker}(T\tilde{E}_{2^j, 2^k})(x, z) \end{aligned}$$

which is $2^{j|\alpha_2| + k|\beta_2|}$ times a term of the original form. Thus, it will suffice to prove the result when $|\alpha_2| = 0 = |\beta_2|$.

Fix $z \in G$. Let us consider the function of x given by:

$$\begin{aligned} & \nabla_{L,x}^{\alpha_1} \nabla_{R,x}^{\beta_1} \text{Ker}(TE_{2^j, 2^k})(x, z) \\ &= 2^{-j|\alpha_2| - k|\beta_2|} \text{Ker} \left(\nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \tilde{E}_{2^j, 2^k} \right) (x, z) \end{aligned}$$

Here α_2 and β_2 are not the same ordered multi-indices as before; rather, we have applied Definition 2.4, and the term on the right hand side of the above equation really denotes a finite sum of such terms. Letting $\phi_{2^j, 2^k}(x) = \text{Ker} \left(\tilde{E}_{2^j, 2^k} \right) (x, z)$, we are considering the function given by:

$$2^{-j|\alpha_2| - k|\beta_2|} \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^j, 2^k}$$

where $\phi_{2^j, 2^k}$ is a $2^j, 2^k$ elementary kernel, and $|\alpha_2|, |\beta_2|$ can be as large as we like.

Theorem 4.11 allows us to create a partition of unity $\psi_l(x)$ ($l \geq 0$), such that:

1. $0 \leq \psi_l \leq 1$, for every l
2. ψ_0 is supported where $\rho_{2^{j-k}}^L(x, z) \lesssim 2^{-j}$
3. ψ_l is supported where $\rho_{2^{j-k}}^L(x, z) \approx 2^{-j+l}$, $l \neq 0$
4. $\left| \nabla_L^\alpha \nabla_R^\beta \phi \right| \lesssim 2^{|\alpha|(j-l) + |\beta|(k-l)}$ and, if $|\alpha| + |\beta| > 0$, is supported where $\rho_{2^{j-k}}^L(x, z) \approx 2^{l-j}$, even if $l = 0$.

This follows from Theorem 4.11 directly for z in the closed unit ball, and for 2^{-j+l} small. However Theorem 4.11 really holds for all points in G and all distances. Creating a bump function of radius r centered at z is equivalent to creating a bump function of radius δ centered at $\frac{\delta}{r}z$. Thus Theorem 4.11 extends to all points and all radii, by homogeneity (just take δ small enough), giving us the above partition of unity.

Define $\phi_l(x) = \psi_l(x) \phi_{2^j, 2^k}(x, z)$ (thinking of z as fixed), so that $\sum_{l \geq 0} \phi_l = \phi_{2^j, 2^k}$. We wish to show that ϕ_l is 2^{-lN} times a $2^{j-l}, 2^{k-l}$ normalized bump function, where N is any integer we choose, and we really mean a constant times a normalized bump function, with support in, perhaps, a constant times the radius of the support it's supposed to have.

We already know that the support of ϕ_l is correct, by the properties of ψ_l , so we turn to estimating derivatives of ϕ_l . When $l > 0$, we have:

$$\begin{aligned} \left| \nabla_L^\alpha \nabla_R^\beta \phi_l \right| &\leq \sum_{\substack{a_1 + a_2 = |\alpha| \\ b_1 + b_2 = |\beta|}} 2^{(a_1(j-l) + b_1(k-l))} \frac{2^{a_2 j + b_2 k}}{2^{lN} V_{2^{j-k}}^L(z, 2^{l-j})} \\ &\lesssim \frac{2^{|\alpha|(j-l) + |\beta|(k-l)}}{2^{lN'} V_{2^{j-k}}^L(z, 2^{l-j})} \end{aligned}$$

here, a_1 represents the number of ∇_L derivatives that land on ψ_l , and a_2 represents the number that land on $\phi_{2^j, 2^k}$. We have used in the last line that we make take N large. This establishes that ϕ_l ($l > 0$) is 2^{-lN} times a $2^j, 2^k$ bump function. When $l = 0$, a nearly identical proof establishes the result.

Thus, by Lemma 7.6, and using the fact that we may take $|\alpha_2|, |\beta_2|$ as large as we like, we have:

$$\begin{aligned}
& 2^{-j|\alpha_2|-k|\beta_2|} \left| \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_l \right| \\
& \lesssim 2^{-j|\alpha_2|-k|\beta_2|-lN} B(2^{j-l}, 2^{k-l}, |\alpha_1| + |\alpha_2|, |\beta_1| + |\beta_2|, m, x, z) \\
& \leq 2^{-lN} B(2^{j-l}, 2^{k-l}, |\alpha_1|, |\beta_1|, m, x, z) \\
& \leq 2^{-lN'} B(2^j, 2^k, |\alpha_1|, |\beta_1|, m, x, z)
\end{aligned}$$

Where we have used in the last step that the function:

$$2^{-l(m+1)} B(2^{j-l}, 2^{k-l}, |\alpha_1|, |\beta_1|, m, x, z)$$

decreases as l increases. Hence,

$$2^{-j|\alpha_2|-k|\beta_2|} \left| \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^j, 2^k} \right| \lesssim B(2^j, 2^k, |\alpha_1|, |\beta_1|, m, x, z)$$

completing the proof. \square

Proposition 7.7 shows that $\text{Ker}(TE_{2^j, 2^k})$ satisfies the growth estimates of a $2^j, 2^k$ elementary kernel. Thus, to show that $TE_{2^j, 2^k}$ is a $2^j, 2^k$ elementary operator, it now remains to show that we may “pull out” derivatives, as in Definition 2.4. To do this, it will suffice to show that the class of operators \mathcal{A} commutes with ∇_L and ∇_R derivatives. By this, we mean:

Theorem 7.8. *Suppose $T \in \mathcal{A}$. Then,*

$$T \nabla_L^\alpha \nabla_R^\beta = \sum_{\substack{|\alpha_1|=|\alpha| \\ |\beta_1|=|\beta|}} \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T_{\alpha_1, \beta_1}$$

where $T_{\alpha_1, \beta_1} \in \mathcal{A}$.

To see why Theorem 7.8 completes the proof that $\mathcal{A} \subseteq \mathcal{A}'$, consider:

$$\begin{aligned}
TE_{2^j, 2^k} &= \sum_{\substack{|\alpha_1|=N_1 \\ |\alpha_2|=N_2 \\ |\beta_1|=N_3 \\ |\beta_2|=N_4}} 2^{-j(N_1+N_2)-k(N_3+N_4)} T \nabla_L^{\alpha_1} \nabla_R^{\beta_1} \tilde{E}_{2^j, 2^k} \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \\
&= \sum 2^{-j(N_1+N_2)-k(N_3+N_4)} \nabla_L^{\alpha_1} \nabla_R^{\beta_1} T_{\alpha_1, \beta_1} \tilde{E}_{2^j, 2^k} \nabla_L^{\alpha_2} \nabla_R^{\beta_2}
\end{aligned}$$

a finite sum of terms satisfying the proper bounds associated to elementary kernels. Hence, we conclude this section by proving Theorem 7.8.

Proof of Theorem 7.8. We will show that if $X_L = \nabla_L^\alpha$, $|\alpha| = 1$, then $TX_L = \sum_{|\alpha_1|=1} \nabla_L^{\alpha_1} T_{\alpha_1}$, and the whole result will follow by symmetry and induction. Let \mathcal{J} be the homogeneous fundamental solution to the sublaplacian:

$$\sum_{|\alpha|=1} (\nabla_L^\alpha)^2$$

See [Fol75] for background on \mathcal{J} . Note that:

$$TX_L = \sum_{|\alpha|=1} (\nabla_L^\alpha)^2 \mathcal{J} TX_L = \sum_{|\alpha|=1} \nabla_L^\alpha ((\nabla_L^\alpha \mathcal{J}) TX_L)$$

Thus, it suffices to show that $STX_L \in \mathcal{A}$, where S is a left invariant convolution operator, with kernel of type 1, in the sense of [Fol75]. Hence, we wish to estimate terms like:

$$\left\langle \phi_{r_L^{(1)}, r_R^{(1)}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} STX_L \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{r_L^{(2)}, r_R^{(2)}}^y \right\rangle_{L^2}$$

where everything above is as in Definition 2.2. However, $\nabla_R^{\beta_1} S = S \nabla_R^{\beta_1}$ and $\nabla_L^{\alpha_1} S = \sum_{|\alpha'_1|=|\alpha_1|} S_{\alpha'_1} \nabla_L^{\alpha'_1}$, where $S_{\alpha'_1}$ is a left invariant operator with convolution kernel of type 1. Thus, it suffices to bound terms of the form:

$$\left\langle \phi_{r_L^{(1)}, r_R^{(1)}}^x, S \nabla_L^{\alpha_1} \nabla_R^{\beta_1} TX_L \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{r_L^{(2)}, r_R^{(2)}}^y \right\rangle_{L^2}$$

where S is an operator with convolution kernel of type 1. (It is easy to see that all the integrals involved converge absolutely, by Lemma 7.6.) Let ψ be a C_0^∞ bump function supported on the unit ball, which is 1 on the ball of radius $3/4$, and $0 \leq \phi \leq 1$. Let $K(x)$ be the convolution kernel of S , and define: $\phi(x) = (\phi(x) - \phi(2x)) K(x)$. Then,

$$S = \sum_{j \in \mathbb{Z}} 2^{-j} \text{Op}_L \left(\phi^{(2^j)} \right)$$

Applying Lemma 6.1, we see:

$$\begin{aligned} & \left| \left\langle \phi_{2^{j_1}, 2^{k_1}}^x, S \nabla_L^{\alpha_1} \nabla_R^{\beta_1} TX_L \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^y \right\rangle_{L^2} \right| \\ & \leq \sum_{j \in \mathbb{Z}} 2^{-j} \left| \left\langle \text{Op}_L \left(\phi^{(2^j)} \right) \phi_{2^{j_1}, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} TX_L \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^y \right\rangle_{L^2} \right| \\ & \lesssim \sum_{j \in \mathbb{Z}} 2^{-j} \left| \left\langle \tilde{\phi}_{2^{j_1} \wedge j, 2^{k_1}}^x, \nabla_L^{\alpha_1} \nabla_R^{\beta_1} TX_L \nabla_L^{\alpha_2} \nabla_R^{\beta_2} \phi_{2^{j_2}, 2^{k_2}}^y \right\rangle_{L^2} \right| \\ & \lesssim 2^{-j} \sum_{j \in \mathbb{Z}} B(2^{j_1 \wedge j \wedge j_2}, 2^{k_1 \wedge k_2}, |\alpha_1| + |\alpha_2| + 1, |\beta_1| + |\beta_2|, x, y) \end{aligned} \tag{33}$$

Let $j_0 = j_1 \wedge j_2$, $k_0 = k_1 \wedge k_2$, $a = |\alpha_1| + |\alpha_2|$, $b = |\beta_1| + |\beta_2|$. Then, we separate the sum on the left hand side of (33) into two sums:

$$\begin{aligned} \sum_{j \geq j_0} 2^{-j} B(2^{j_0}, 2^{k_0}, a+1, b, m, x, y) &= 2^{-j_0} B(2^{j_0}, 2^{k_0}, a+1, b, m, x, y) \\ &= B(2^{j_0}, 2^{k_0}, a, b, m, x, y) \end{aligned}$$

and,

$$\begin{aligned} \sum_{j \leq j_0} 2^{-j} B(2^j, 2^{k_0}, a+1, b, m, x, y) &= \sum_{j \leq j_0} B(2^j, 2^{k_0}, a, b, m, x, y) \\ &\approx B(2^{j_0}, 2^{k_0}, a, b, m, x, y) \end{aligned}$$

where we have used that we may take a and b large, and we have applied Lemma 7.1. This completes the proof. \square

Remark 7.9. Theorem 7.8 is the only place where we use the crucial hypothesis that we have a cancellation condition that happens on *both* sides of T at once.

8 L^p Boundedness

In this section, we show that operators in \mathcal{A}' extend to bounded operators on L^p , $1 < p < \infty$. To do this, we will need a relevant Littlewood-Paley square function, and a relevant maximal function. Fortunately, we will be able to construct both out of the building blocks of the analogous operators for left and right convolution operators.

We begin with the maximal functions. Define:

$$\begin{aligned} (\mathcal{M}_L(f))(x) &= \sup_{R>0} \frac{1}{R^Q} \int_{\|y^{-1}x\| < R} |f(y)| dy \\ &= \sup_{R>0} \text{Op}_L \left(\chi_B^{(\frac{1}{R})} \right) |f| \end{aligned}$$

and similarly,

$$\mathcal{M}_R(f) = \sup_{R>0} \text{Op}_R \left(\chi_B^{(\frac{1}{R})} \right) |f|$$

It follows from the results in [Ste93] that

$$\|\mathcal{M}_L f\|_{L^p(G)} \lesssim \|f\|_{L^p(G)}$$

for $1 < p < \infty$, and similarly for \mathcal{M}_R . For us, the relevant maximal function will be:

$$\mathcal{M}f = \sup_{R_1>0, R_2>0} \text{Op}_L \left(\chi_B^{(\frac{1}{R_1})} \right) \text{Op}_R \left(\chi_B^{(\frac{1}{R_2})} \right) |f|$$

It is easy to see that

$$\mathcal{M}f \leq \mathcal{M}_L \mathcal{M}_R f \tag{34}$$

and therefore,

$$\|\mathcal{M}f\|_{L^p(G)} \lesssim \|f\|_{L^p(G)}$$

Corresponding to each $1 \geq \epsilon > 0$, we get a maximal function for ρ_ϵ^L (and one for ρ_ϵ^R , but let's focus on ρ_ϵ^L), defined by:

$$\mathcal{M}_\epsilon f(x) = \sup_{R>0} \frac{1}{V_\epsilon^L(x, R)} \int_{\rho_\epsilon^L(x, y) \leq R} |f(y)| dy$$

But, then, taking χ as in Remark 5.4, we see:

$$\mathcal{M}_\epsilon f \lesssim \sup_{R>0} \text{Op}_L \left(\chi^{(\frac{1}{R})} \right) \text{Op}_R \left(\chi^{(\frac{1}{\epsilon R})} \right) |f| \lesssim \mathcal{M}f$$

so we see that \mathcal{M} uniformly bounds the maximal functions corresponding to all of the geometries we are considering.

Lemma 8.1. *Suppose $\text{Ker}(E_{r_L, r_R})$ satisfies the bounds (3) of Definition 2.4 without any derivatives (ie it is like an elementary kernel, but we need not be able to “pull out” derivatives or take derivatives). Then, for $f \in \mathcal{S}$, $|E_{r_L, r_R} f| \lesssim \mathcal{M}f$, with constants uniform in all the relevant parameters.*

Proof. We prove this in the case $r_L = 2^j$, $r_R = 2^k$, the more general case following from this one. We assume $k \geq j$, the other case following in the same manner. Consider,

$$\begin{aligned} |E_{2^j, 2^k} f(x)| &\lesssim \int_{\rho_{2^j-k}^L(x, y) \leq 2^{-j}} \frac{|f(y)|}{V_{2^{k-j}}^L(x, 2^{-j})} dy \\ &\quad + \sum_{l \geq 1} \int_{\rho_{2^j-k}^L(x, y) \approx 2^{l-j}} \frac{|f(y)|}{2^{lN} V_{2^{k-j}}^L(x, 2^{l-j})} dy \\ &\leq \sum_{l \geq 0} 2^{-lN} \mathcal{M}_{2^{k-j}} f \\ &\lesssim \mathcal{M}f \end{aligned}$$

completing the proof. \square

Recall the definition $\mathcal{S}_{(m)}$. $\phi \in \mathcal{S}_{(m)}$ if and only if $\phi = \sum_{|\alpha|=m} \nabla_L^\alpha \psi_\alpha$, where $\psi \in \mathcal{S}$. We get essentially the same space if we replace ∇_L by ∇_R (by that we mean $\mathcal{S}_{(m_1)}^R \subset \mathcal{S}_{(m)}^L \subset \mathcal{S}_{(m_2)}^R$ where $m_1, m_2 \rightarrow \infty$ as $m \rightarrow \infty$). In short, being a high order of ∇_L derivatives is the same as being a high order of ∇_R derivatives, which is the same as moments up to a high order vanishing. We have:

Lemma 8.2. *For any $N \in \mathbb{N}$ there exist function $\phi_1, \dots, \phi_M, \psi_1, \dots, \psi_M \in \mathcal{S}_{(N)}$ (here M depends on N) such that:*

$$\sum_{l=1}^M \sum_{j \in \mathbb{Z}} \phi_l^{(2^j)} * \psi_l^{(2^j)} = \delta_0$$

Proof. This follows directly from Theorem 1.61 in [FS82]. \square

We will be able to use the ψ_l and ϕ_l from Lemma 8.2 to construct a relevant Littlewood-Paley square function. Henceforth, we fix such ψ_l and ϕ_l , thinking of N as large (how large N will have to be will be implicit in our proof).

We define

$$\begin{aligned}\Lambda_{j,k}^{l_1,l_2} &= \text{Op}_L \left(\phi_{l_1}^{(2^j)} \right) \text{Op}_R \left(\psi_{l_2}^{(2^k)} \right) \\ P_{j,k}^{l_1,l_2} &= \text{Op}_L \left(\psi_{l_1}^{(2^j)} \right) \text{Op}_R \left(\phi_{l_2}^{(2^k)} \right)\end{aligned}$$

so that,

$$\sum_{l_1,l_2} \sum_{j,k \in \mathbb{Z}} P_{j,k}^{l_1,l_2} \Lambda_{j,k}^{l_1,l_2} = I$$

and we define our square function:

$$\Lambda(f) = \left(\sum_{l_1,l_2} \sum_{j,k \in \mathbb{Z}} \left| \Lambda_{j,k}^{l_1,l_2} f \right|^2 \right)^{\frac{1}{2}}$$

Theorem 8.3. For $1 < p < \infty$,

$$\|f\|_{L^p(G)} \approx \|\Lambda(f)\|_{L^p(G)}$$

Proof. Fix l_1, l_2 (recall, l_1 and l_2 just range over a finite set). The theorem will follow if we can show that for any sequence of 1s and -1 s, $\epsilon_{j,k}$, we have that

$$\sum_{j,k} \epsilon_{j,k} \Lambda_{j,k}^{l_1,l_2}$$

is bounded on L^p , uniformly in the choice of the sequence $\epsilon_{j,k}$ (and with a similar result for the $P_{j,k}^{l_1,l_2}$, which will follow in the same way). To see why this is enough, see p. 267 of [Ste93] and Chapter 4, Section 5 of [Ste70].

However,

$$\begin{aligned}\sum_{j,k} \epsilon_{j,k} \Lambda_{j,k}^{l_1,l_2} &= \sum_{j,k} \text{Op}_T \left(\epsilon_{j,k} \phi_{l_1}^{(2^j)}(x) \psi_{l_2}^{(2^k)}(y) \right) \\ &= \text{Op}_T \left(\sum_{j,k} \epsilon_{j,k} \phi_{l_2}^{(2^j)}(x) \psi_{l_2}^{(2^k)}(y) \right)\end{aligned}$$

and

$$\sum_{j,k} \epsilon_{j,k} \phi_{l_1}^{(2^j)}(x) \psi_{l_2}^{(2^k)}(y)$$

converges to a product kernel, uniformly in the choice of $\epsilon_{j,k}$ (see [NRS01], Theorem 2.2.1). Hence, Corollary 3.14 shows us that

$$\sum_{j,k} \epsilon_{j,k} \Lambda_{j,k}^{l_1,l_2}$$

is uniformly bounded on L^p . □

Theorem 8.4. *Suppose $T \in \mathcal{A}'$. Then, T extends to a bounded operator $L^p \rightarrow L^p$, $1 < p < \infty$.*

Proof. We first prove the result imagining that $\phi_l, \psi_l \in \mathcal{S}_0$. At the end, we will explain why it is enough to have them in $\mathcal{S}_{(N)}$ for some large N . This proof is more or less standard, however we include it to help make clear where we are using $\phi_l, \psi_l \in \mathcal{S}_{(N)}$.

Since $\phi_l, \psi_l \in \mathcal{S}_0$, we have (by Theorem 6.2) that $\Lambda_{j,k}^{l_1,l_2}, P_{j,k}^{l_1,l_2}$ are $2^j, 2^k$ elementary kernels (uniformly in j, k). Hence, we have (for $f \in \mathcal{S}$):

$$\begin{aligned} \Lambda_{j_1,k_1}^{l_1,l_2} T P_{j_2,k_2}^{l'_1,l'_2} f &= \Lambda_{j_1,k_1}^{l_1,l_2} E_{2^{j_2}, 2^{k_2}} \\ &= 2^{-|j_1-k_1|-|j_2-k_2|} E_{2^{j_1}, 2^{k_1}} f \\ &\lesssim 2^{-|j_1-k_1|-|j_2-k_2|} \mathcal{M} f \end{aligned}$$

where we have used the definition of \mathcal{A}' , Lemma 6.3, and Lemma 8.1, and $E_{2^j, 2^k}$ just represents some $2^j, 2^k$ elementary kernel that may change from line to line.

Define $F_{j,k}^{l_1,l_2} = \Lambda_{j,k}^{l_1,l_2} T f$. Then,

$$\begin{aligned} |F_{j,k}^{l_1,l_2}| &= \left| \sum_{l'_1, l'_2} \sum_{j_1, k_1} \Lambda_{j,k}^{l_1,l_2} T P_{j_1,k_1}^{l'_1,l'_2} \Lambda_{j_1,k_1}^{l'_1,l'_2} f \right| \\ &\lesssim \sum_{l'_1, l'_2} 2^{-|j_1-j|-|k_1-k|} \mathcal{M} \left(\Lambda_{j_1,k_1}^{l'_1,l'_2} f \right) \end{aligned}$$

and hence,

$$\begin{aligned} |F_{j,k}^{l_1,l_2}|^2 &\lesssim \left[\sum_{l'_1, l'_2} \sum_{k_1, j_1} 2^{-|j_1-j|-|k_1-k|} \left(\mathcal{M} \left(\Lambda_{j_1,k_1}^{l'_1,l'_2} f \right) \right)^2 \right] \left[\sum_{l'_1, l'_2} \sum_{j_1, k_1} 2^{-|j_1-j|-|k_1-k|} \right] \\ &\approx \sum_{l'_1, l'_2} \sum_{k_1, j_1} 2^{-|j_1-j|-|k_1-k|} \left(\mathcal{M} \left(\Lambda_{j_1,k_1}^{l'_1,l'_2} f \right) \right)^2 \end{aligned}$$

and so, we have

$$\sum_{l_1, l_2} \sum_{j, k} |F_{j,k}^{l_1,l_2}|^2 \lesssim \sum_{l_1, l_2} \sum_{j, k} \left(\mathcal{M} \left(\Lambda_{j,k}^{l_1,l_2} f \right) \right)^2$$

Putting all of this together, we see:

$$\begin{aligned}
\|Tf\|_{L^p(G)} &\approx \left\| \left(\sum_{\ell_1, \ell_2} \sum_{j, k} |F_{j, k}^{\ell_1, \ell_2}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \\
&\lesssim \left\| \left(\sum_{\ell_1, \ell_2} \sum_{j, k} \left(\mathcal{M} \left(\Lambda_{j, k}^{\ell_1, \ell_2} f \right) \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \\
&\lesssim \|\Lambda(f)\|_{L^p(G)} \\
&\approx \|f\|_{L^p(G)}
\end{aligned}$$

where we have used the vector valued maximal function, see [Ste93], Chapter 2, Section 1. The vector valued inequality comes from (34), and the corresponding inequalities for $\mathcal{M}_L, \mathcal{M}_R$ as shown in [Ste93].

Now we turn to explaining why we only need $\phi_l, \psi_l \in \mathcal{S}_{(N)}$ for some fixed large N . Indeed, this proof used only a finite number of the semi-norms that define the elementary operators. This follows from the fact that every proof we have done about elementary operators was continuous. For example Lemma 6.3 showed:

$$E_{2^{j_1}, 2^{k_1}} E_{2^{j_2}, 2^{k_2}} = 2^{-|j_1 - j_2| - |k_1 - k_2|} \tilde{E}_{2^{j_1}, 2^{k_1}}$$

Where each semi-norm of \tilde{E} was bounded in terms of a finite number of semi-norms of the terms on the left hand side. Thus, this proof only required a finite number of semi-norms, which we may control by taking N large. The only potential worry is the line where we used the definition of \mathcal{A}' (ie, T takes elementary operators to elementary operators); since this was a definition, and we do not have a priori continuity in the above sense. However, this continuity follows from a combination of Theorem 6.6 and Lemma 6.3. \square

9 Pseudolocality

In this section, we show that the operators in \mathcal{A}' are pseudolocal, and calculate bounds for derivatives of the kernel away from the diagonal; although we will not put these bounds in a closed form. In Section 9.1, however, we will derive a closed form for the growth of the kernel off the diagonal in the case that G is the three dimensional Heisenberg group.

Fix $T \in \mathcal{A}'$. Decompose T as in Theorem 6.6:

$$T = \sum_{j, k \in \mathbb{Z}} E_{2^j, 2^k}$$

We will imagine this is a finite sum, and show that the result is uniformly C^∞ off of the diagonal. It will then follow that T is pseudolocal. In fact, we will

prove the bounds separately for

$$T_L = \sum_{k \geq j} E_{2^j, 2^k}$$

$$T_R = \sum_{j \geq k} E_{2^j, 2^k}$$

We focus on T_L , the bounds for T_R being the same, with the roles of right and left reversed. Set $K(x, z) = \text{Ker}(T_L)$.

Let us return to the notation from Section 2:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_n$$

It is easy to see that each vector field ∇_R^α with $|\alpha| = 1$ can be written in the form:

$$\sum_{j=1}^n q_{j,\alpha}(x) X_{L,\alpha,j} \quad (35)$$

where $X_{L,\alpha,j} \in V_j$ (when thought of as left invariant vector fields), and q is a homogeneous polynomial of degree $j-1$. For example, on the Heisenberg group, $X_R = X_L - 4y\partial_t$ (see Section 9.1 for this notation).

Lemma 9.1. *Let $\phi(x, z)$ be a $2^j, 2^k$ elementary kernel (we are still assuming $k \geq j$). Then, if $|\alpha_1| + |\alpha_2| = a$, $|\beta_1| + |\beta_2| = b$,*

$$\left| \nabla_{L,x}^{\alpha_1} \nabla_{R,x}^{\beta_1} \nabla_{L,z}^{\alpha_2} \nabla_{R,z}^{\beta_2} \phi(x, z) \right|$$

$$\lesssim \frac{2^{ja+kb} \left(1 \wedge \left(\sum_{|\alpha|=1} \sum_{s=1}^n |q_{s,\alpha}(x)| 2^{sj-k} \right) \right)}{(1 + 2^j \rho_{2^{j-k}}^L(x, z))^N V_{2^{j-k}}^L(x, 2^{-j} + \rho_{2^{j-k}}^L(x, z))}$$

where $N \geq 0$ is fixed and as large as we like.

Proof. It is easy reduce to the case when $a = 0 = b$, just from the definition of an elementary kernel. The case when $1 = 1 \wedge \left(\sum_{|\alpha|=1} \sum_{s=1}^n |q_{s,\alpha}(x)| 2^{sj-k} \right)$ follows directly from the definition of an elementary kernel. For the other case, consider (in what follows, ψ with any subscript will denote a $2^j, 2^k$ elementary operator):

$$\begin{aligned} \phi(x, z) &= \sum_{|\beta|=1} 2^{-k} \nabla_{R,x}^\beta \psi_\beta(x, z) \\ &= \sum_{|\beta|=1} \sum_{s=1}^n 2^{-k} q_{s,\beta}(x) X_{L,\beta,s} \psi_\beta(x, z) \\ &= \sum_{|\beta|=1} \sum_{s=1}^n 2^{sj-k} q_{s,\beta}(x) \psi_{s,\beta}(x, z) \end{aligned}$$

now the claim follows by taking absolute values and applying the definition of elementary kernels. \square

Theorem 9.2. Let $a = |\alpha_1| + |\alpha_2|$. Then, for $x \neq z$, we have

$$\begin{aligned} & \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} K(x,z) \right| \\ & \lesssim \sum_{l=0}^{\infty} \rho_{2^{-l}}^L(x,z)^{-a} \left(\frac{1}{V_{2^{-l}}^L(x, \rho_{2^{-l}}^L(x,z))} \wedge \frac{\sum_{|\alpha|=1} \sum_{s=1}^n |q_{s,\alpha}(x)| \rho_{2^{-l}}^L(x,z)^{-s+1}}{2^l V_{2^{-l}}^L(x, \rho_{2^{-l}}^L(x,z))} \right) \end{aligned}$$

and therefore T_L is pseudolocal. A similar result holds for T_R , thereby showing that T is pseudolocal.

Proof. First let's see why this shows that T_L is pseudolocal. We claim that the above sum converges absolutely. This can be seen by using the facts that $\rho_\epsilon^L \geq \rho_1^L$ and $V_\epsilon^L \geq V_0^L$ (for all $\epsilon \in [0, 1]$) and thus if we only take the right part of the \wedge we get a geometric series. Hence, the whole series converges absolutely, showing that T_L is pseudolocal.

Let $\phi_{2^j, 2^k} = \text{Ker}(E_{2^j, 2^k})$. Let $0 \leq l = k - j$, and to save space, define $\delta_l = \rho_{2^{-l}}^L(x, z)$. We think of $l \geq 0$ as fixed, and sum over all those k, j such that $k - j = l$. Using Lemma 9.1, we see:

$$\begin{aligned} & \left| \nabla_{L,x}^{\alpha_1} \nabla_{L,z}^{\alpha_2} \sum_{k-j=l} \phi_{2^j, 2^k}(x, z) \right| \\ & \lesssim \sum_{k-j=l} 2^{ja} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| 2^{j(s-1)})}{(1 + 2^j \delta_l)^N V_{2^{-l}}^L(x, 2^{-j} + \delta_l)} \\ & = \sum_{j \in \mathbb{Z}} 2^{ja} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| 2^{j(s-1)})}{(1 + 2^j \delta_l)^N V_{2^{-l}}^L(x, 2^{-j} + \delta_l)} \end{aligned}$$

where, in the numerator, $\sum = \sum_{|\alpha|=1} \sum_{s=1}^n$, and we have suppressed the x in $q_{s,\alpha}(x)$. We separate the above sum into two sums: when $2^j \geq \frac{1}{\delta_l}$ and when $2^j \leq \frac{1}{\delta_l}$. Now,

$$\begin{aligned} \sum_{2^j \geq \frac{1}{\delta_l}} & \approx \sum_{2^j \geq \frac{1}{\delta_l}} 2^{ja} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| 2^{j(s-1)})}{(2^j \delta_l)^N V_{2^{-l}}^L(x, \delta_l)} \\ & \approx \delta_l^{-a} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| \delta_l^{(1-s)})}{V_{2^{-l}}^L(x, \delta_l)} \end{aligned}$$

since the second term is a geometric sum (when N is sufficiently large), and therefore bounded by its first term. This is precisely the bound we were striving for. We now turn to the sum when $2^j \leq \frac{1}{\delta_j}$:

$$\begin{aligned} \sum_{2^j \leq \frac{1}{\delta_j}} & \approx \sum_{2^j \leq \frac{1}{\delta_j}} 2^{ja} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| 2^{j(s-1)})}{V_{2^{-l}}^L(x, 2^{-j})} \\ & \approx \delta_l^{-a} \frac{1 \wedge (2^{-l} \sum |q_{s,\alpha}| \delta_l^{(1-s)})}{V_{2^{-l}}^L(x, \delta_l)} \end{aligned}$$

where this follows since the above sum is geometric, and therefore bounded by its first term, completing the proof. \square

Remark 9.3. It seems likely that Theorem 9.2 is the best we can do (at least when $a = 0$). Indeed, if K_1 and K_2 are Calderón-Zygmund kernels, we can decompose:

$$\begin{aligned} K_1 &= \sum_j \nabla_L \cdot \phi_j^{(2^j)} \\ K_2 &= \sum_k \nabla_R \cdot \psi_k^{(2^k)} \end{aligned}$$

where ϕ_j, ψ_k are d -tuples of C_0^∞ functions supported on the unit ball B (see [NRS01]). Then we consider:

$$\text{Op}_L(K_1) \text{Op}_R(K_2) = \sum_{j,k} \text{Op}_L \left(\nabla_L \cdot \phi_j^{(2^j)} \right) \text{Op}_R \left(\nabla_R \cdot \psi_k^{(2^k)} \right)$$

One wishes to use the fact that $\nabla_L \cdot \phi_j$ and $\nabla_R \cdot \phi_k$ are derivatives of functions to yield a gain over the estimate given in Theorem 5.2. The standard way of doing this, when $k \geq j$, is to integrate the ∇_R by parts over to ϕ_j . However, this process is exactly the one we used in Lemma 9.1.

The observant reader will note, however, that the bound in Theorem 9.2 is not actually symmetric in x and z , as the optimal bound should be. And that, moreover, we could use the same proof to prove a seemingly better symmetric bound. This turns out to not be an essential point, and indeed the bound is essentially symmetric in x and z . This is exemplified in Section 9.1 in the case of the Heisenberg group. Thus, without some new idea, one is unable to do better than Theorem 9.2.

9.1 The Heisenberg Group

In this section, we derive a closed form for the bound in Theorem 9.2, in the case of the three dimensional Heisenberg group, \mathbb{H}^1 . As a manifold $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$, and we give it coordinates $(z, t) = (x, y, t)$. The multiplication is given by $(z, t)(w, s) = (z + w, t + s + 2\text{Im}(z\overline{w}))$. The dilation is given by $r(z, t) = (rz, r^2t)$. The left invariant vector fields of order 1 are spanned by $X_L = \partial_x + 2y\partial_t$, $Y_L = \partial_y - 2x\partial_t$, while the right invariant vector fields of order 1 are spanned by $X_R = \partial_x - 2y\partial_t$, $Y_R = \partial_y + 2x\partial_t$. We also have:

$$[X_L, Y_L] = -4\partial_t = -[X_R, Y_R]$$

and so ∂_t spans the left (and right) invariant vector fields of order 2.

We fix $(z, t), (w, s) \in \mathbb{H}^1$, $(z, t) \neq (w, s)$, and we again define (for $l \geq 0$), $\delta_l = \rho_{2^{-l}}^L((z, t), (w, s))$. Note that $\delta_\infty = \rho_0^L$, $\delta_0 = \rho_1^L$. Fix $\alpha \in \mathbb{N}$, $\alpha \geq 0$ (α will play the role of a in Theorem 9.2), and let $\zeta = (z, t)$. We will show:

Theorem 9.4.

$$\sum_{l=0}^{\infty} \delta_l^{-\alpha} \left(\frac{1}{V_{2^{-l}}^L(\zeta, \delta_l)} \wedge \frac{1 + |z| \delta_l^{-1}}{2^l V_{2^{-l}}^L(\zeta, \delta_l)} \right) \approx \frac{1}{\delta_{\infty}^2 \delta_0^{2+\alpha}} \quad (36)$$

Note that this sum is exactly the one that appears in Theorem 9.2, in the case of \mathbb{H}^1 .

The first question to address is: When do we use each side of the \wedge ? Namely, we are interested in the question, when is $\rho_{2^{-l}}^L = \delta_l \gtrsim 2^{-l} |z|$? So let us investigate the question: When is $\epsilon |z| \lesssim \rho_{\epsilon}^L$? The answer is that it is true precisely when $\rho_{\epsilon}^L \approx \rho_0^L$. Indeed, suppose we are on the scale $\rho_{\epsilon}^L \approx \delta$, here δ is just some number > 0 , not to be confused with δ_l . (We assume $\delta < 1$, and that we are working very close to 0 and then extend the results by homogeneity.) Then, by Theorem 4.8, we wish to find the maximal determinant among 3×3 submatrices of:

$$\begin{bmatrix} \delta & 0 & -2y\delta \\ 0 & \delta & 2x\delta \\ 0 & 0 & \delta^2 \\ \delta\epsilon & 0 & 2y\delta\epsilon \\ 0 & \delta\epsilon & -2x\delta\epsilon \\ 0 & 0 & \delta^2\epsilon^2 \end{bmatrix}$$

Now the largest three determinants are given by: Rows $(1, 2, 3) = \delta^4$, $(1, 2, 4) = 4\delta^3 y \epsilon$, and $(1, 2, 5) = -4\delta^3 x \epsilon$. Thus, when $\delta \geq |z| \epsilon$, $(1, 2, 3)$ is the largest determinant (up to a constant), and therefore on this scale $\rho_{\epsilon}^L \approx \rho_0^L$ (since the first 3 rows corresponded to the left invariant vector fields).

Remark 9.5. Actually, this proof extends to an arbitrary stratified group. That is, the right part of the \wedge in Theorem 9.2 is less than the left part precisely when $\rho_{2^{-l}}^L \approx \rho_0^L$. We leave the details to the interested reader.

Proposition 9.6. Suppose $\rho_{\epsilon}^L((z, t), (w, s)) \lesssim |z| \epsilon$. Then,

$$\rho_{\epsilon}^L((z, t), (w, s)) \approx |z - w| + \frac{1}{\epsilon |z|} |t - s + 2\text{Im}(z\bar{w})|$$

Proof. When $\epsilon = 1$, the result follows easily. Moreover, if $1 \geq \epsilon \geq \frac{1}{2}$, we have $\rho_{\epsilon}^L \approx \rho_1^L$, and the result follows from the case when $\epsilon = 1$. Henceforth, we restrict our attention to the case $\epsilon < \frac{1}{2}$. Fix $\delta \lesssim |z| \epsilon$. We will show that the following conditions are equivalent:

1. $\rho_{\epsilon}^L((z, t), (w, s)) \lesssim \delta$.
2. $\exists w_0, s_0$ such that $|w - w_0| \lesssim \delta$, $|s_0 - s + 2\text{Im}(w_0\bar{w})| \lesssim \delta^2$, $|z - w_0| \lesssim \epsilon \delta$, $|t - s_0| \lesssim \epsilon |z| \delta$.
3. $\exists s_0$ such that $|z - w| \lesssim \delta$, $|s_0 - s + 2\text{Im}(z\bar{w})| \lesssim \delta^2$, $|t - s_0| \lesssim \delta |z| \epsilon$.
4. $\exists s_0$ such that $|z - w| \lesssim \delta$, $|s_0 - s + 2\text{Im}(z\bar{w})| \lesssim \delta |z| \epsilon$, $|t - s_0| \lesssim \delta |z| \epsilon$.

5. $|z - w| \lesssim \delta, |t - s + 2\operatorname{Im}(z\overline{w})| \lesssim \delta |z| \epsilon.$
6. $|z - w| + \frac{1}{\epsilon|z|} |t - s + 2\operatorname{Im}(z\overline{w})| \lesssim \delta.$

this will complete the proof, since the statement of the proposition is $1 \Leftrightarrow 6$. We will show the equivalences in the following order:

$$1 \Leftrightarrow 2, \quad 3 \Rightarrow 2 \Rightarrow 4 \Rightarrow 3, \quad 4 \Leftrightarrow 5, \quad 5 \Leftrightarrow 6$$

To see that $1 \Leftrightarrow 2$, we apply Theorem 4.13 to see:

$$\rho_\epsilon^L = (\epsilon \rho_0^R) \circ (\epsilon \rho_0^L) \circ ((1 - \epsilon) \rho_0^L) = (\epsilon \rho_1^L) \circ ((1 - \epsilon) \rho_0^L) \approx (\epsilon \rho_1^L) \circ \rho_0^L$$

where we have used that $\epsilon < \frac{1}{2}$, and the obvious fact that $(\epsilon \rho_0^L) \circ ((1 - \epsilon) \rho_0^L) = \rho_0^L$. The statement $1 \Leftrightarrow 2$ now follows directly from the definition of $(\epsilon \rho_1^L) \circ \rho_0^L$. For $3 \Rightarrow 2$ merely take $w_0 = z$. Suppose we have 2. Consider,

$$\begin{aligned} |z - w| &\leq |z - w_0| + |w_0 - w| \lesssim \delta + \epsilon \delta \lesssim \delta \\ |s_0 - s + 2\operatorname{Im}(z\overline{w})| &\leq |2\operatorname{Im}((z - w_0)\overline{w})| + |s_0 - s + 2\operatorname{Im}(w_0\overline{w})| \\ &\lesssim \delta^2 + |\operatorname{Im}((z - w_0)\overline{w})| \\ &= \delta^2 + |\operatorname{Im}((z - w_0)(\overline{w} - w_0 + z))| \\ &\leq \delta^2 + |z - w_0| |w - w_0| + |z - w_0| |z| \\ &\lesssim \delta^2 + \epsilon \delta^2 + |z| \epsilon \delta \\ &\lesssim |z| \epsilon \delta \end{aligned}$$

and we therefore have $2 \Rightarrow 4$. Suppose we have 4. Define $s_1 = s - 2\operatorname{Im}(z\overline{w})$. We will show that 3 holds with s_1 in place of s_0 (the notation s_0 has already been used in the definition of 4). Indeed,

$$|s_1 - s + 2\operatorname{Im}(z\overline{w})| = 0 \lesssim \delta^2$$

$$|t - s_1| = |t - s + 2\operatorname{Im}(z\overline{w})| \leq |s_0 - s + 2\operatorname{Im}(z\overline{w})| + |s_0 - t| \lesssim |z| \delta \epsilon$$

and so 3 holds.

$5 \Rightarrow 4$ follows just by taking $s_0 = t$. Suppose we have 4, consider:

$$|t - s + 2\operatorname{Im}(z\overline{w})| \leq |s_0 - s + 2\operatorname{Im}(z\overline{w})| + |t - s_0| \lesssim |z| \delta \epsilon$$

and so 5 holds. Finally, $6 \Leftrightarrow 5$ is obvious. \square

We now claim that the condition $\epsilon |z| \lesssim \rho_\epsilon^L$ is the same as the condition $\epsilon |z| \lesssim \rho_0^L$. Indeed, $\rho_\epsilon^L \leq \rho_0^L$, and so one direction is clear. For the other direction, fix $\eta > 0$ and consider those ϵ such that $\epsilon |z| \geq \eta \rho_\epsilon^L$. The left hand side decreases to 0 as ϵ decreases, while the right hand side increases. Thus there is a least ϵ (call it ϵ_0) for which it holds. For this ϵ_0 , we have $\epsilon_0 |z| = \eta \rho_{\epsilon_0}^L$, so by the above remarks, we have $\rho_{\epsilon_0}^L \approx \rho_0^L$, and thus, $\epsilon_0 |z| \geq C \rho_0^L$. Hence for all ϵ such that $\epsilon |z| \geq \eta \rho_\epsilon^L$ (namely all $\epsilon \geq \epsilon_0$), we have $\epsilon |z| \geq C \rho_0^L$.

We separate the sum (36) into two parts: when $2^l \delta_\infty \leq |z|$ and when $2^l \delta_\infty \geq |z|$. We first look at the case when $2^l \delta_\infty \leq |z|$, so that by Proposition 9.6, we have

$$\delta_l \approx |z - w| + 2^l \frac{|t - s + 2\operatorname{Im}(z\bar{w})|}{|z|}$$

and we see by Theorem 4.9 and the remarks above that

$$V_{2^{-l}}^L(\zeta, \delta_l) \approx 2^{-l} |z| \delta_l^3$$

to save space, we denote $b = |z - w|$ and $a = \frac{|t-s+2\operatorname{Im}(z\bar{w})|}{|z|}$, so that

$$\delta_l \approx b + 2^l a$$

Also, let $c = \frac{|z|}{\delta_\infty}$. Thus, we consider:

$$\begin{aligned} & \sum_{2^l \delta_\infty \leq |z|} \delta_l^{-\alpha} \left(\frac{1}{V_{2^{-l}}^L(\zeta, \delta_l)} \wedge \frac{1 + |z| \delta_l^{-1}}{2^l V_{2^{-l}}^L(\zeta, \delta_l)} \right) \\ & \approx \sum_{2^l \delta_\infty \leq |z|} \delta_l^{-\alpha} \frac{1}{V_{2^{-l}}^L(\zeta, \delta_l)} \\ & \approx \sum_{2^l \leq c} \frac{2^l}{|z| (b + 2^l a)^{3+\alpha}} \\ & \approx \frac{1}{|z| a^{3+\alpha}} \sum_{2^l \leq c} \frac{2^l}{\left(\frac{b}{a} + 2^l\right)^{3+\alpha}} \\ & \approx \frac{1}{|z| a^{3+\alpha}} \int_0^{\log_2(c)} \frac{2^t}{\left(\frac{b}{a} + 2^t\right)^{3+\alpha}} dt \\ & \approx \frac{1}{|z| a^{3+\alpha}} \int_1^c \frac{1}{\left(\frac{b}{a} + u\right)^{3+\alpha}} du \\ & \approx \frac{1}{|z| a^{3+\alpha}} \frac{\left(c + \frac{b}{a}\right)^{2+\alpha} - \left(1 + \frac{b}{a}\right)^{2+\alpha}}{\left(c + \frac{b}{a}\right)^{2+\alpha} \left(1 + \frac{b}{a}\right)^{2+\alpha}} \end{aligned}$$

Recall, $l \geq 0$ and this sum is nonzero only when $c \geq 2^l$. In fact, let us ignore the possibility that this sum is nonzero when $c \leq 2$ (this case can be taken care of in a similar manner to our sum when $c \leq 2^l$). Thus, we have:

$$\begin{aligned} & \approx \frac{1}{|z| a^{3+\alpha}} \frac{c \left(\sum_{j=1}^{2+\alpha} \left(\frac{b}{a}\right)^{2+\alpha-j} c^{j-1} \right)}{\left(c + \frac{b}{a}\right)^{2+\alpha} \left(1 + \frac{b}{a}\right)^{2+\alpha}} \\ & \approx \frac{1}{|z| a^{3+\alpha}} \frac{c \left(c + \frac{b}{a}\right)^{2+\alpha-1}}{\left(c + \frac{b}{a}\right)^{2+\alpha} \left(1 + \frac{b}{a}\right)^{2+\alpha}} \end{aligned}$$

$$\begin{aligned}
&\approx \frac{1}{|z| a^{3+\alpha}} \frac{c}{\left(c + \frac{b}{a}\right) \left(1 + \frac{b}{a}\right)^{2+\alpha}} \\
&= \frac{1}{\delta_\infty \left(\frac{|z|}{\delta_\infty} a + b\right) (a+b)^{2+\alpha}} \\
&\approx \frac{1}{\delta_\infty \left(\frac{|z|}{\delta_\infty} a + b\right) \delta_0^{2+\alpha}}
\end{aligned}$$

where, in the last line, we have used that $(a+b) \approx \delta_0$, by Proposition 9.6. Finally, we will be done with this sum provided we can show

$$\frac{|z|}{\delta_\infty} a + b \approx \delta_\infty$$

To see this, consider ϵ_0 such that $\epsilon_0 |z| = \rho_{\epsilon_0}^L$ (as before). Then, for this ϵ_0 , we have $\epsilon_0 |z| = \rho_{\epsilon_0}^L \approx \rho_0^L$. However, we also have $\epsilon_0 |z| \gtrsim \rho_{\epsilon_0}^L$, and therefore,

$$\delta_\infty = \rho_0^L \approx \rho_{\epsilon_0}^L \approx \left(b + \frac{a}{\epsilon_0}\right) \approx \left(b + \frac{|z|}{\rho_0^L} a\right) = \left(b + \frac{|z|}{\delta_\infty} a\right)$$

completing the proof for this sum.

We now turn to the sum when $2^l \delta_\infty \geq |z|$. In this case, $\delta_l \approx \delta_\infty$, and $V_{2^{-l}}^L(\zeta, \delta_l) \approx V_0^L(\zeta, \delta_\infty)$, by the remarks at the beginning of the proof. And thus, we are considering:

$$\begin{aligned}
&\sum_{2^l \delta_\infty \geq |z|} \delta_l^{-\alpha} \left(\frac{1}{V_{2^{-l}}^L(\zeta, \delta_l)} \wedge \frac{1 + |z| \delta_l^{-1}}{2^l V_{2^{-l}}^L(\zeta, \delta_l)} \right) \\
&\approx \sum_{2^l \delta_\infty \geq |z|} \delta_\infty^{-\alpha} \left[\frac{|z|}{2^l \delta_\infty V_0^L(\zeta, \delta_\infty)} + \frac{1}{2^l V_0^L(\zeta, \delta_\infty)} \right] \\
&\approx \sum_{2^l \delta_\infty \geq |z|} \frac{|z|}{2^l \delta_\infty^{5+\alpha}} + \frac{1}{2^l \delta_\infty^{4+\alpha}}
\end{aligned}$$

the first sum is geometric, and therefore bounded by its first term, and we have:

$$\approx \frac{1}{\delta_\infty^{4+\alpha}} + \sum_{2^l \delta_\infty \geq |z|} \frac{1}{2^l \delta_\infty^{4+\alpha}}$$

The second term above is bounded by:

$$\sum_{l \geq 0} \frac{1}{2^l \delta_\infty^{4+\alpha}} \approx \frac{1}{\delta_\infty^{4+\alpha}}$$

Hence, the whole sum is:

$$\approx \frac{1}{\delta_\infty^{4+\alpha}}$$

Since $\delta_\infty \geq \delta_0$, this completes the proof.

10 Closing Remarks

Definition 2.2 only tested high derivatives of the operator T . One can replace Definition 2.2 with an equivalent definition that works for derivatives of all orders, but with the price that the B must be quite a bit more complicated. Let $q_{j,\alpha}^R$ be the functions from (35) and $q_{j,\alpha}^L$ be the corresponding ones with the roles of left and right reverse. We then define:

$$\begin{aligned} & \tilde{B}(2^{j_0}, 2^{k_0}, x, y, N_L, N_R) \\ &= \sum_{j \leq j_0, k \leq k_0} 2^{jN_L + kN_R} \left(1 \wedge \left(\sum_{|\alpha|=1} \sum_{s=1}^N |q_{s,\alpha}^R(x)| 2^{sj-k} \right) \wedge \left(\sum_{|\alpha|=1} \sum_{s=1}^N |q_{s,\alpha}^L(x)| 2^{sk-j} \right) \right) \\ & \quad \times K_{2^j, 2^k}(x, y) \end{aligned}$$

where $K_{2^j, 2^k}$ is the function from Section 5.1. Note that \tilde{B} does not involve m . Then Definition 2.2 with B replaced by \tilde{B} now works for all $N_L, N_R \geq 0$, and defines the same algebra. That these algebras are the same follow in a manner similar to the bounds in the rest of this paper.

One may think of the operators in \mathcal{A} and \mathcal{A}' as “smoothing of order 0.” To come up with an analogous definition for operators which are “smoothing of order s_L ” in the left invariant vector fields and “smoothing of order s_R ” in the right invariant vector fields, it suffices to modify Definitions 2.2 and 2.5 only slightly.

Indeed, we say an operator $T : \mathcal{S}_{(N)} \rightarrow \mathcal{S}'$ (for some large N) is in \mathcal{A}_{s_L, s_R} if it satisfies the conditions of Definition 2.2 with $B(\cdot, \cdot, N_L, N_R, \cdot, \cdot, \cdot)$ replaced by $B(\cdot, \cdot, N_L - s_L, N_R - s_R, \cdot, \cdot, \cdot)$.

We say an operator $T : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ is in \mathcal{A}'_{s_L, s_R} if $r_L^{s_L} r_R^{s_R} T E_{r_L, r_R}$ is an r_L, r_R elementary operator for every r_L, r_R elementary operator E_{r_L, r_R} , uniformly in the relevant parameters. Then, $\mathcal{A}_{s_L, s_R} = \mathcal{A}'_{s_L, s_R}$ (under the obvious identification). It is clear that if $T_1 \in \mathcal{A}'_{s_L^1, s_R^1}, T_2 \in \mathcal{A}'_{s_L^2, s_R^2}$, then we have $T_1 T_2 \in \mathcal{A}'_{s_L^1 + s_L^2, s_R^1 + s_R^2}$, and therefore we have a similar result for \mathcal{A}_{s_L, s_R} (remember, we are just thinking of these operators on $\mathcal{S}_{(N)}$ for N large). Many of the results of this paper extend to these operators in the obvious way.

Finally, let us consider the question of whether or not it is really necessary to have a cancellation condition on both sides simultaneously as in Definition 2.2, as opposed to something more along the lines of the standard definitions of Calderón-Zygmund operators. One could think about this in two ways. One could try to use a one sided cancellation condition along the lines of Lemma 7.6 (or something slightly stronger, in terms of the \tilde{B} above), along with a growth condition of the kernel of T off of the diagonal. However, in light of Theorem 9.4, any condition along these lines seems likely to be necessarily weaker than our Definition 2.2.

Alternatively, let us go back to considering the composition of

$$\text{Op}_L(K_1) \text{Op}_R(K_2)$$

where K_1, K_2 are Calderón-Zygmund kernels. We decompose

$$K_1 = \sum_{j \in \mathbb{Z}} \phi_j^{(2^j)}$$

$$K_2 = \sum_{k \in \mathbb{Z}} \psi_k^{(2^k)}$$

where ϕ_j, ψ_k form a bounded subset of C_0^∞ , are supported in the unit ball, and have mean 0 (see [NRS01], Theorem 2.2.1). This cancellation condition on the ϕ_j essentially tells us that if η is another C_0^∞ function supported on the unit ball, we have:

$$\text{Op}_L(K_1) \text{Op}_L(\eta^{(2^{j_0})}) = \sum_{j \leq j_0} \text{Op}_L(\tilde{\phi}_j^{(2^j)})$$

where the $\tilde{\phi}_j$ are essentially of the same form as the ϕ_j , and with a similar result for $\text{Op}_L(\eta^{(2^{j_0})}) \text{Op}_L(K)$. Thus, if η_1, η_2 are of the same form as η , we have:

$$\text{Op}_L(\eta_1^{(2^{j_1})}) \text{Op}_L(K_1) \text{Op}_R(\eta_2^{(2^{j_2})}) = \sum_{j \leq j_1 \wedge j_2} \text{Op}_L(\tilde{\phi}_j^{(2^j)})$$

Hence, for composition $\text{Op}_L(K_1) \text{Op}_R(K_2)$ we want:

$$\begin{aligned} & \text{Op}_L(\eta_1^{(2^{j_1})}) \text{Op}_R(\eta_2^{(2^{k_1})}) \text{Op}_L(K_1) \text{Op}_R(K_2) \text{Op}_L(\eta_3^{(2^{j_2})}) \text{Op}_R(\eta_4^{(2^{k_2})}) \\ &= \sum_{\substack{j \leq j_1 \wedge j_2 \\ k \leq k_1 \wedge k_2}} \text{Op}_L(\tilde{\phi}_j^{(2^j)}) \text{Op}_R(\tilde{\psi}_k^{(2^k)}) \end{aligned}$$

and so a cancellation condition on one side alone will be fine in the case when $k_1 \leq k_2$ and $j_1 \leq j_2$ (or the reverse situation), but seems like it will not be able to yield the desired estimate when $k_1 < k_2$ and $j_2 < j_1$ (or the reverse situation).

References

- [DF79] John D. Dollard and Charles N. Friedman, *Product integration with applications to differential equations*, Encyclopedia of Mathematics and its Applications, vol. 10, Addison-Wesley Publishing Co., Reading, Mass., 1979, With a foreword by Felix E. Browder, With an appendix by P. R. Masani.
- [DJS85] G. David, J.-L. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana **1** (1985), no. 4, 1–56.

- [Fol75] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), no. 2, 161–207.
- [FS82] G. B. Folland and Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982.
- [GAV89] R. V. Gamkrelidze, A. A. Agrachëv, and S. A. Vakhrameev, *Ordinary differential equations on vector bundles, and chronological calculus*, Current problems in mathematics. Newest results, Vol. 35 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, Translated in J. Soviet Math. **55** (1991), no. 4, 1777–1848, pp. 3–107.
- [Kis95] Vladimir V. Kisil, *Connection between two-sided and one-sided convolution type operators on non-commutative groups*, Integral Equations Operator Theory **22** (1995), no. 3, 317–332.
- [Koe02] Kenneth D. Koenig, *On maximal Sobolev and Hölder estimates for the tangential Cauchy-Riemann operator and boundary Laplacian*, Amer. J. Math. **124** (2002), no. 1, 129–197.
- [Koh05] J. J. Kohn, *Hypoellipticity and loss of derivatives*, Ann. of Math. (2) **162** (2005), no. 2, 943–986, With an appendix by Makhlouf Derriidj and David S. Tartakoff.
- [NRS01] Alexander Nagel, Fulvio Ricci, and Elias M. Stein, *Singular integrals with flag kernels and analysis on quadratic CR manifolds*, J. Funct. Anal. **181** (2001), no. 1, 29–118.
- [NS01] Alexander Nagel and Elias M. Stein, *Differentiable control metrics and scaled bump functions*, J. Differential Geom. **57** (2001), no. 3, 465–492.
- [NS06] ———, *The $\bar{\partial}_b$ -complex on decoupled boundaries in \mathbb{C}^n* , Ann. of Math. (2) **164** (2006), no. 2, 649–713.
- [NSW85] Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), no. 1-2, 103–147.
- [Ste70] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
- [Ste93] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

- [Str07] Brian Street, *A parametrix for kohn's operator*, Ph.D. thesis, Princeton University, 2007.
- [Trè67] François Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.